# GOING UP OF THE $u$-INVARIANT OVER FORMALLY REAL FIELDS

CLAUS SCHUBERT

Abstract. Let $F$ be a field of characteristic not 2, and assume $F$ has finite reduced stability. Let $K/F$ be any finite extension. We prove that if the general $u$-invariant $u(F)$ is finite, then $u(K)$ is finite.

## 1. Introduction

Let $F$ be a field of characteristic not 2. We denote by $WF$ the Witt ring of $F$, i.e., the ring of equivalence classes of non-degenerate quadratic forms over $F$. Its torsion part is denoted by $W_tF$. The (general) $u$-invariant of $F$ is defined to be

$$u(F) := \max\{\dim \varphi_{an} \mid \varphi \in W_tF\}$$

or $u(F) = \infty$ if no such maximum exists. We are interested in the behavior of the $u$-invariant under finite field extensions $K/F$. We say $u$ "goes up" a field extension $K/F$ if finiteness of $u(F)$ implies finiteness of $u(K)$, and $u$ "goes down" if finiteness of $u(K)$ implies finiteness of $u(F)$. If $F$ is non-formally real, then $u(F) < \infty$ implies that $u(K) < \infty$ ([EL76, Theorem F]). If $K = F(\sqrt{w})$ is a quadratic extension where $w \in F^\times$ is totally positive, then $u(F) < \infty$ if and only if $u(K) < \infty$ ([EL76, Theorem H]). If $K/F$ is a finite normal extension, then $u(K) < \infty$ implies that $u(F) < \infty$ ([Elm77, Theorem 3.2]). In particular, the finiteness of the $u$-invariant goes down arbitrary quadratic extensions ([Elm77, Theorem 3.1]). However, in general, Going Up will not hold for arbitrary quadratic extensions when the base field is formally real. For example, the field $F = \mathbb{R}((x_1))((x_2))((x_3))\dots$ of iterated Laurent series in infinitely many variables over $\mathbb{R}$ has $u$-invariant $u(F) = 0$, but $F(\sqrt{-1}) = \mathbb{C}((x_1))((x_2))((x_3))\dots$ has $u$-invariant $u(F(\sqrt{-1})) = \infty$. Thus, for Going Up, additional assumptions will be needed. We will show in this article that assuming that $F$ has finite reduced stability is sufficient and necessary.

In this article, a form over $F$ means a non-degenerate quadratic form. In the proof of the main result we use results from the theory of abstract spaces of orderings and apply them to the theory of quadratic forms over fields. The terminology and results on spaces of orderings can be found in [Mar96]. We collect some of the basic definitions and results here.

Let $G$ be an elementary 2-group, written multiplicatively, with a distinguished element $-1$. A *form* $\varphi$ over $G$ is a symbol $\varphi = \langle a_1, \dots, a_n\rangle$ with $a_i \in G$ for all $i$. If $\varphi = \langle a_1, \dots, a_n\rangle$ and $\psi = \langle b_1, \dots, b_m\rangle$ are forms over $G$, then the *sum*

*Date*: February 1, 2007.
2000 *Mathematics Subject Classification.* 11E04, 11E81.
This article is based on part of the author's Ph.D. thesis, written under the supervision of Richard Elman.

of $\varphi$ and $\psi$ is $\varphi \oplus \psi := \langle a_1, \ldots, a_n, b_1, \ldots, b_m\rangle$. The *character group* of $G$ is $\chi(G) := \mathrm{Hom}(G, \{-1, 1\})$.

Let $X$ be a nonempty set and $G$ a subgroup of $\{-1, 1\}^X$. Let $\varphi = \langle a_1, \ldots, a_n\rangle$ be a form over $G$. Then the *dimension* of $\varphi$ is $n$ and the *signature* of $\varphi$ at $\alpha \in X$ is the integer $\mathrm{sgn}_\alpha(\varphi) := \sum_{i=1}^n a_i(\alpha)$. Two forms $\varphi$ and $\psi$ are *isometric (over $X$)*, write $\varphi \cong \psi$, if they have the same dimension and the same signature at each $\alpha \in X$. We say $\varphi$ *represents* $a \in G$ if $\varphi \cong \langle a, a_2, \ldots, a_n\rangle$ with $a_2, \ldots, a_n \in G$. The set of all elements of $G$ represented by $\varphi$ is denoted by $D(\varphi)$. A *space of orderings* is a pair $(X, G)$ satisfying the following axioms:

(1) $X$ is a nonempty set, $G$ is a subgroup of $\{-1, 1\}^X$ containing the constant function $-1$, and $G$ separates points in $X$ (i.e., if $x \neq y \in X$ then there exists $g \in G$ such that $g(x) \neq g(y)$).

(2) The image of the embedding $X \to \chi(G)$, $x \mapsto (g \mapsto g(x))$ is closed in $\chi(G)$. We will usually identify $X$ with its image in $\chi(G)$ from here on.

(3) If $\varphi$ and $\psi$ are forms over $G$ and $x \in D(\varphi \oplus \psi)$ then there are $y \in D(\varphi)$ and $z \in D(\psi)$ such that $x \in D(\langle y, z\rangle)$.

In particular, the ordering space $X_F$ of a formally real field $F$ is a space of orderings $(X, G)$ in the above sense with $X = X_F$ and $G$ given by the group of generalized square classes $F^\times/\sigma(F)$, where $\sigma(F) = \{x_1^2 + \ldots + x_n^2 \mid x_i \in F^\times, i = 1, \ldots, n\}$ ([Mar80b, Theorem 1.3]).

The elements of $X$ are called *orderings*. For an element $a \in G$ we define $H(a) = \{\alpha \in X \mid \alpha(a) = 1\}$, the *Harrison set* of $a$. We define a topology on $X$ with subbasis consisting of all Harrison sets. $X$ is compact with this topology. A *subspace* of a space of orderings is a subset $Y \subset X$ closed under linear combinations of orderings, i.e., if $\alpha_1, \ldots, \alpha_n \in Y$, then $\alpha_1\alpha_2 \ldots \alpha_n \in Y$. A subspace of a space of orderings is itself a space of orderings. We say $(X, G)$ is the *direct sum* of $(X_i, G_i)$, $i \in I$, write $(X, G) = \bigoplus_{i \in I}(X_i, G_i)$, if $X$ is the disjoint union of the $X_i$ and $G$ consists of all functions $g \in \{-1, 1\}^X$ satisfying $g|_{X_i} \in G_i$, $i \in I$. The *translation group* of $X$ is $gr(X) := \{\gamma \in \chi(G) \mid \gamma X = X\}$. If $gr(X) \neq 1$, then we define the *residue space* of $X$ as $(X', G')$ where $G' = gr(X)^\perp := \{a \in G \mid \gamma(a) = 1 \text{ for all } \gamma \in gr(X)\}$ is a subgroup of $G$ and $X'$ denotes the image of $X$ in $\chi(G')$ via restriction. In this case (or more generally, if $G'$ is any subgroup of $G$), the space $(X, G)$ is called a *group extension* of $(X', G')$. We call $X$ an *elementary indecomposable space (EI-space)* if either $|X| = 1$ or $gr(X) \neq 1$ and $|X| \geq 4$.

A form $\varphi$ over $G$ is called *isotropic* if $\varphi \cong \langle 1, -1, a_3, \ldots, a_n\rangle$ with $a_3, \ldots, a_n \in G$ and *anisotropic* otherwise. Just as in the classical theory of quadratic forms over fields, every form $\varphi$ over $G$ decomposes as $\varphi \cong i\langle 1, -1\rangle \oplus \varphi_{an}$ where $\varphi_{an}$ is anisotropic and $i \geq 0$ an integer. Two forms $\varphi$ and $\psi$ are *Witt equivalent* if $\varphi_{an} \cong \psi_{an}$. The equivalence classes of forms over $G$ under this equivalence relation form a ring called the *(abstract) Witt ring* of $(X, G)$ and denoted by $W(X)$. The cokernel of the map $W(X) \to C(X, \mathbb{Z})$ sending $\varphi$ to $\widehat{\varphi} : X \to \mathbb{Z}$, $\widehat{\varphi}(\alpha) = \mathrm{sgn}_\alpha \varphi$ is 2-primary torsion (cf., for example, [Mar80b, Lemma 5.4]). Here $\mathbb{Z}$ is endowed with the discrete topology. The *stability* $\mathrm{st}(X)$ of a space of orderings $(X, G)$ is defined to be the minimal 2-power that annihilates this cokernel, or $\infty$ if no such minimum exists. If $X_F$ is the ordering space of a formally real field $F$, then $\mathrm{st}(X_F)$ is called the *reduced stability* of $F$, denoted by $\mathrm{st}_r F$. By [Brö74, Satz 3.17], $\mathrm{st}_r F \leq n$ if and only if $I^{n+1}F = 2I^nF + I_t^{n+1}F$ where $I^nF$ is the $n$-th power of the fundamental ideal $IF$ in the Witt ring $WF$ and $I_t^nF = I^nF \cap W_tF$. For convenience, we define the reduced stability of a non-formally real field to be 0. The main structure results we use for spaces of orderings $(X, G)$ are summarized below.

**Fact 1** ([Mar80a, Theorem 2.6 and Remark 2.7]). *If* $\operatorname{st}(X) < \infty$ *then the connected components* $X_i$, $i \in I$ *of* $X$ *are EI-subspaces and* $W(X)$ *consists of all the* $\varphi \in C(X, \mathbb{Z})$ *satisfying* $\varphi|_{X_i} \in W(X_i)$ *and* $\operatorname{sgn}_\alpha \varphi \equiv \operatorname{sgn}_\beta \varphi \pmod 2$ *for all* $\alpha \in X_i$, $\beta \in X_j$, $i \neq j$.

**Fact 2** ([Mar80a, Remark 2.7]). *If* $(X, G)$ *has components* $(X_i, G_i)$, $i \in I$, *and* $\operatorname{st}(X) \neq 1$ *is finite, then* $\operatorname{st}(X) = \max\{\operatorname{st}(X_i) : i \in I\}$.

**Fact 3.** *If* $\operatorname{st}(X) < \infty$ *and* $(X, G)$ *is a group extension of* $(X', G')$ *then* $[G : G'] < \infty$. *If* $[G : G'] = 2^k$ *then* $\operatorname{st}(X) = \operatorname{st}(X') + k$.

Fact 3 follows from [Mar80b, Theorem 6.4] and an easy induction argument.

## 2. Matching Signatures over Spaces of Orderings

Let $(X, G)$ be a space of orderings and $\varphi$ a form over $G$. We wish to find a form $\psi$ over $G$ satisfying $\operatorname{sgn}_\alpha \varphi = \operatorname{sgn}_\alpha \psi$ for all $\alpha \in X$ and with $\psi$ having minimal possible dimension. The easiest case occurs when $\operatorname{st}(X) \leq 1$. In this case, we say $(X, G)$ satisfies the *strong approximation property (SAP)*. It is easily proved that if $(X, G)$ satisfies SAP and $A$ and $B$ are disjoint, closed subsets of $X$, then there exists $a \in G$ such that $\alpha(a) = -1$ for $\alpha \in A$ and $\alpha(a) = 1$ for $\alpha \in B$ (cf. [Brö74, Satz 3.20] for the field case; the same proof works in the general case).

**Lemma 1.** *Let* $(X, G)$ *be a space of orderings satisfying SAP. Let* $\varphi$ *be a form over* $G$ *with* $|\operatorname{sgn}_\alpha \varphi| \leq m$ *for all* $\alpha \in X$. *Then there exists a form* $\psi$ *over* $G$ *of dimension at most* $m$ *such that* $\operatorname{sgn}_\alpha(\varphi - \psi) = 0$ *for all* $\alpha \in X$.

*Proof.* We induct on $m$. If $m = 0$, there is nothing to prove. For $m > 0$, let $\varphi$ be a form over $G$ satisfying $|\operatorname{sgn}_\alpha \varphi| \leq m$ for all $\alpha \in X$ and let $A = \{\alpha \in X \mid \operatorname{sgn}_\alpha \varphi \geq 0\}$ and $B = X \backslash A$. Note that both $A$ and $B$ are clopen (closed and open). Since $X$ satisfies SAP, there exists $a \in G$ such that $\alpha(a) = -1$ for all $\alpha \in A$ and $\alpha(a) = 1$ for all $\alpha \in B$. Then $|\operatorname{sgn}_\alpha(\varphi \oplus \langle a \rangle)| \leq m - 1$ for all $\alpha \in X$, so by the induction assumption there exists a form $\widetilde{\psi}$ over $G$ such that $\dim \widetilde{\psi} \leq m - 1$ and $\operatorname{sgn}_\alpha(\varphi \oplus \langle a \rangle - \widetilde{\psi}) = 0$ for all $\alpha \in X$. Then $\psi = \widetilde{\psi} - \langle a \rangle$ works. □

Our aim is to give a bound on the dimension of forms $\psi$ as above where $(X, G)$ has finite stability. By Fact 1, all the components of $X$ are EI-subspaces. Thus, we will have to examine what happens to signatures of forms over a group extension $X$ of a space $X'$. The following lemma deals with this case.

**Lemma 2.** *Let* $(X, G)$ *be a space of orderings that is a group extension of* $(X', G')$ *with* $|G/G'| = 2$, *say* $G = G' \vee xG'$. *Let* $\varphi$ *be a form over* $G$ *with* $|\operatorname{sgn}_\alpha \varphi| \leq n$ *for all* $\alpha \in X$. *Then there are forms* $\varphi_1, \varphi_2$ *over* $G'$ *such that* $\varphi = \varphi_1 \oplus x\varphi_2$ *and* $|\operatorname{sgn}_\alpha \varphi_1| + |\operatorname{sgn}_\alpha \varphi_2| \leq n$ *for all* $\alpha \in X$. *In particular,* $|\operatorname{sgn}_\alpha \varphi_i| \leq n$ *for* $i = 1, 2$ *and all* $\alpha \in X$.

*More generally, suppose that* $|G/G'| = 2^k$, *say* $G/G'$ *is generated by* $\{x_1, \ldots, x_k\}$. *Let* $\varphi$ *be a form over* $G$ *with* $|\operatorname{sgn}_\alpha \varphi| \leq n$ *for all* $\alpha \in X$. *Then there are forms* $\varphi_\varepsilon$ *over* $G'$ *with* $|\operatorname{sgn}_\alpha \varphi_\varepsilon| \leq n$ *for all* $\alpha \in X$ *such that*

$$\varphi = \bigoplus_{\varepsilon} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \varphi_\varepsilon$$

*where the sum is taken over all possible* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$, $\varepsilon_i \in \{0, 1\}$.

*Proof.* Assume that $|G/G'| = 2$, say $G = G' \vee xG'$. Let $\varphi$ be a form over $G$. Then we can write $\varphi \cong \langle a_1, \dots, a_m, xb_1, \dots, xb_k\rangle$ with $a_i, b_i \in G'$ for all $i$. Let $\varphi_1 = \langle a_1, \dots, a_m\rangle$ and $\varphi_2 = \langle b_1, \dots, b_k\rangle$. Then $\varphi = \varphi_1 \oplus x\varphi_2$ and $\varphi_1$, $\varphi_2$ are forms over $G'$. Let $\alpha \in X$ be any ordering in $X$, and let $\beta \in X$ be the corresponding ordering such that $\beta|_{X'} = \alpha|_{X'}$ and $\beta(x) = -\alpha(x)$. Without loss of generality, assume that $\alpha(x) = 1$. Let $r = \operatorname{sgn}_\alpha \varphi_1$ and $s = \operatorname{sgn}_\alpha \varphi_2 = \operatorname{sgn}_\alpha x\varphi_2$. If $r$ and $s$ are both positive or both negative, then $|\operatorname{sgn}_\alpha \varphi| = |r + s| \leq n$ and so $|r| + |s| \leq n$. If $r$ and $s$ have opposite signs, then note that $r = \operatorname{sgn}_\beta \varphi_1$ and $-s = \operatorname{sgn}_\beta x\varphi_2$, so by the first case, $|r| + |-s| \leq n$.

Now assume $|G/G'| = 2^k$ and $G/G'$ is generated by $\{x_1, \dots, x_k\}$. We successively construct subgroups $G_i$, $i = 0, \dots, k$, such that $G = G_0$ and $G' = G_k$ and $G_i = G_{i+1} \vee x_{i+1}G_{i+1}$, where $i = 0, \dots, k-1$. By using the case $k = 1$ above and induction on $k$, it is clear that we may write $\varphi = \bigoplus_\varepsilon x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \varphi_\varepsilon$ with forms $\varphi_\varepsilon$ over $G'$, where the sum is taken over all possible $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_i \in \{0, 1\}$. The fact that each $\varphi_\varepsilon$ has signature $|\operatorname{sgn}_\alpha \varphi_\varepsilon| \leq n$ for all $\alpha \in X$ follows again from the above case $k = 1$ and induction, and from the fact that for any form $\varphi$ over $G'$ and any $\alpha \in X$, we have $\operatorname{sgn}_\alpha \varphi = \operatorname{sgn}_{\alpha|_{X'}} \varphi$. □

**Remark 1.** The signature of a form over $G$ is invariant under equivalence of spaces of orderings in the following sense: if $\varphi = \langle a_1, a_2, \dots, a_k\rangle$ and $\alpha \in X$, then $\operatorname{sgn}_\alpha \varphi = \sum_{i=1}^k \alpha(a_i)$. Now assume there is an equivalence of spaces $(X', G') \sim (X, G)$, i.e., an isomorphism $f : G \xrightarrow{\sim} G'$ together with its dual isomorphism $f^* : \chi(G') \xrightarrow{\sim} \chi(G)$ such that $f^*(X') = X$. Then for $\alpha' \in X'$, the form $\varphi' = \langle f(a_1), f(a_2), \dots, f(a_k)\rangle$ has signature

$$\operatorname{sgn}_{\alpha'} \varphi' = \sum_{i=1}^k \alpha'(f(a_i)) = \sum_{i=1}^k f^*(\alpha')(a_i) = \operatorname{sgn}_{f^*(\alpha')} \varphi.$$

Thus, if $\psi$ is a form over $G$ such that $\operatorname{sgn}_\alpha(\varphi - \psi) = 0$ for all $\alpha \in X$, then there exists a form $\psi'$ over $G'$ of the same dimension such that $\operatorname{sgn}_{\alpha'}(\varphi' - \psi') = 0$ for all $\alpha' \in X'$. Hence any result that we obtain on the relationship between signatures and dimensions of forms on an ordering space $(X, G)$ will be true for any equivalent space $(X', G')$.

We now formulate our result on signature matching of forms over ordering spaces with finite stability. This provides the crucial part in the proof of the main result of this article.

**Theorem 1.** *Let* $(X, G)$ *be a space of orderings of stability* $\operatorname{st} X = n < \infty$*. Let* $\varphi$ *be a form over* $G$ *with* $|\operatorname{sgn}_\alpha \varphi| \leq m$ *for all* $\alpha \in X$*. Then there exists a form* $\psi$ *over* $G$ *such that* $\dim \psi \leq 2^{n-1}m$ *(or* $m$ *if* $n \leq 1$*) and* $\operatorname{sgn}_\alpha(\varphi - \psi) = 0$ *for all* $\alpha \in X$*.*

*Proof.* We induct on $\operatorname{st} X$. If $X$ has stability 0 or 1, then $X$ satisfies SAP. We are done by Lemma 1. For the induction step, we let $(X, G)$ be a space of orderings with $\operatorname{st} X = n > 1$ and assume that the statement holds for all spaces of orderings with stability less than $n$. We consider two cases.

**Case 1.** $X$ is an EI-space.

As $\operatorname{st} X > 1$, we have $|X| > 1$. Thus, by the definition of EI-space, $gr(X) \neq 1$ and $(X, G)$ is a group extension of its residue space $(X', G')$. By Fact 3, $|G/G'| = 2^k$ for some $k$ and $\operatorname{st} X' = n - k$. Here $k > 0$ since otherwise $gr(X) = 1$.

Let $\varphi$ be a form over $G$ with signature $|\operatorname{sgn}_\alpha \varphi| \le m$ for all $\alpha \in X$. By Lemma 2, we may write $\varphi = \bigoplus_\varepsilon x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \varphi_\varepsilon$ where the sum is taken over all possible $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_i \in \{0,1\}$ and where $\varphi_\varepsilon$ are forms over $G'$ with signature $|\operatorname{sgn}_\alpha \varphi_\varepsilon| \le m$ for all $\alpha \in X$ (and hence for all $\alpha \in X'$).

Thus by the induction assumption, for each $\varepsilon$ there is a form $\psi_\varepsilon$ over $G'$ with $\dim \psi_\varepsilon \le 2^{n-k-1} m$ (or $m$ if $|X'| = 1$) and $\operatorname{sgn}_\alpha(\varphi_\varepsilon - \psi_\varepsilon) = 0$ for all $\alpha \in X'$, and hence also for all $\alpha \in X$. Let $\psi = \bigoplus_\varepsilon x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} \psi_\varepsilon$ where again the sum is taken over all possible $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ with $\varepsilon_i \in \{0,1\}$. Then $\operatorname{sgn}_\alpha(\varphi - \psi) = 0$ for all $\alpha \in X$ and $\dim \psi \le 2^k 2^{n-k-1} m = 2^{n-1} m$.

**Case 2.** $X$ is not an EI-space.

In this case, the connected components of $(X, G)$ form a partition of $X$ by [Mar80a, Theorem 2.3]. By Fact 1, the components are EI-subspaces $(X_i, G_i)$ for some $i \in I$. The group $G$ is identified with a subgroup of the direct product $\prod_{i \in I} G_i$. Since $\operatorname{st}(X) = n > 1$, by Fact 2, we have $\operatorname{st}(X) = \max\{\operatorname{st}(X_i) \mid i \in I\}$. We denote by $\pi_i$, $i \in I$, the projection of $\prod_{i \in I} G_i$ on the $i$-th coordinate $G_i$.

Let $\varphi = \langle a_1, a_2, \dots, a_r \rangle$ be a form over $G$ with $|\operatorname{sgn}_\alpha \varphi| \le m$ for all $\alpha \in X$. For $i \in I$ and $j = 1, \dots, r$, let $a_{ij} = \pi_i(a_j) \in G_i$, and let $\varphi_i = \langle a_{i1}, a_{i2}, \dots, a_{ir} \rangle$. Then we have $\operatorname{sgn}_\alpha \varphi_i = \operatorname{sgn}_\alpha \varphi$ for all $\alpha \in X_i$. The spaces $X_i$ have stability less than or equal to $n$ for all $i \in I$ and are EI-spaces. Thus, by the induction assumption and by case 1, there are forms $\psi_i$ of dimension $k_i \le 2^{n-1} m$ over $G_i$ such that $\operatorname{sgn}_\alpha(\varphi_i - \psi_i) = 0$ for all $\alpha \in X_i$, say, $\psi_i = \langle b_{i1}, b_{i2}, \dots, b_{ik_i} \rangle$, $i \in I$. Let $k = \max\{k_i \mid i \in I\}$. If $k_i < k$ for some $i$, replace $\psi_i$ by the form $\langle b_{i1}, b_{i2}, \dots b_{ik_i} \rangle \oplus \frac{k-k_i}{2} \langle 1_{G_i}, -1_{G_i} \rangle$. This does not change the signature of $\psi_i$. So we may assume that $\dim \psi_i = k \le 2^{n-1} m$ for $i \in I$. Let $\psi' = \langle b_1, b_2, \dots, b_k \rangle$ be the form over $\prod_{i \in I} G_i$ with $\pi_i(b_j) = b_{ij}$ for all $i \in I$ and $j = 1, \dots, k$. Then $\psi' \in W(X)$ by Fact 1. Pick a form $\psi$ over $G$ that represents $\psi'$ with $\dim \psi \le \dim \psi'$ (e.g., $\psi$ anisotropic will satisfy this). Then $\operatorname{sgn}_\alpha(\varphi - \psi) = \operatorname{sgn}_\alpha(\varphi_i - \psi_i)$ for all $\alpha \in X_i$ and all $i \in I$, so $\operatorname{sgn}_\alpha(\varphi - \psi) = 0$ for all $\alpha \in X$ and $\dim \psi \le k \le 2^{n-1} m$. $\square$

We will apply this result to the field case in the next section.

## 3. The $u_m(F)$-Invariant

**Definition 1.** Let $F$ be a formally real field. For $m \in \mathbb{N}$, we define

$$u_m(F) := \max_{\varphi \in WF} \left\{ \dim \varphi_{an} \ : \ |\operatorname{sgn}_\alpha \varphi| \le m \text{ for all } \alpha \in X_F \right\}$$

(or $u_m(F) = \infty$ if no such maximum exists).

If $F$ is non-formally real, we define

$$u_m(F) := u(F) \text{ for all } m \in \mathbb{N}.$$

We now apply Theorem 1 to the field case.

**Proposition 1.** *Let $F$ be a field with $\operatorname{st}_r F = n < \infty$. Then for all $m \in \mathbb{N}$,*

$$u_m(F) \le u(F) + 2^{n-1} m$$

*(or $u_m(F) \le u(F) + m$ if $\operatorname{st}_r F = 0$).*

*Proof.* If $F$ is non-formally real there is nothing to prove, so assume $F$ is formally real. Let $\varphi$ be an anisotropic form over $F$ with $|\operatorname{sgn}_\alpha \varphi| \le m$ for all $\alpha \in X_F$. We may view $\varphi$ as a form $\overline{\varphi}$ over $F^\times/\sigma(F)$. By Theorem 1, there exists a form $\overline{\psi} = \langle \overline{a}_1, \dots, \overline{a}_k \rangle$ over $G$ such that $\dim \overline{\psi} \le 2^{n-1} m$ (or $m$ if $n \le 1$) and $\operatorname{sgn}_\alpha(\overline{\varphi} -$

$\overline{\psi}) = 0$ for all $\alpha \in X_F$. We may view $\overline{\psi}$ as a form $\psi = \langle a_1, \dots, a_k\rangle$ over $F$ by picking any representative $a_i \in F^\times$ of its generalized square class $\overline{a}_i$ in $G$ for $i = 1, \dots, k$. By Pfister's Local-Global Principle, the form $\varphi - \psi$ is a torsion form, so $\dim(\varphi - \psi)_{an} \le u(F)$. Hence

$$\dim \varphi \le \dim(\varphi - \psi)_{an} + \dim \psi \le u(F) + 2^{n-1}m.$$

□

**Example 1.** The upper bound in Proposition 1 can be attained. For example, let $F = \mathbb{R}((x))((y))$ and $\varphi_k = k\langle 1, x, y, -xy\rangle$ for $k \ge 1$. Here $n = 2$ and $u(F) = 0$. For each $k$, the form $\varphi_k$ is anisotropic and totally indefinite, has dimension $4k$, and $|\operatorname{sgn}_\alpha \varphi_k| \le 2k$ for all $\alpha \in X_F$. Thus, for all $m = 2k$, we have $u_m(F) \ge 2m$.

## 4. Going Up of the $u$-Invariant

In this section, we show that for any field $F$ with $\operatorname{st}_r F < \infty$, the finiteness of $u(F)$ implies the finiteness of $u(F(\sqrt{-1}))$ and hence the finiteness of $u(K)$ for any finite extension $K/F$. For the proof, we need the following lemma.

**Lemma 3** ([EL72, Theorem 3.2])**.** *Let $F$ be a formally real field with ordering space $X_F$. Let $A$ and $B$ be disjoint closed subsets of $X$. Then there exists a positive integer $n$ and a form $\varphi \in I^nF$ such that $\operatorname{sgn}_\alpha \varphi = 0$ for all $\alpha \in A$ and $\operatorname{sgn}_\alpha \varphi = 2^n$ for all $\alpha \in B$. Furthermore, if $F$ has reduced stability $\operatorname{st}_r F \le k$, we may choose $n = k$.*

The last statement in Lemma 3 follows from [EL72, Proposition 3.7] with only minor modifications.

**Theorem 2.** *Let $F$ be a field with $\operatorname{st}_r F = n - 1 < \infty$, $n \ge 1$. Then*

$$u(F(\sqrt{-1})) \le \frac{1}{2}u(F) + u_{2^n-1}(F).$$

*Proof.* If $F$ is non-formally real then $u_{2^n-1}(F) = u(F)$ and the result follows by [EL76, Theorem 7.1]. So we may assume $F$ is formally real. Let $K = F(\sqrt{-1})$ and let $\psi \in W_tK = WK$ be anisotropic. Then by [Elm77, Theorem 2.1] we can write $\psi \cong \varphi_K \perp \psi_1$ where $\varphi$ is a form over $F$ and $\psi_1$ is a form over $K$ with $s_*(\psi_1) \in W_tF$ anisotropic (where $s_*$ is the transfer induced by the $F$-linear functional $s(\sqrt{-1}) = 1$ and $s(1) = 0$). Thus, $\dim \psi_1 \le \frac{1}{2}u(F)$.

For $m \in \mathbb{N}$, let

$$X_m = \{\alpha \in X_F \mid m2^n \le |\operatorname{sgn}_\alpha \varphi| < (m+1)2^n\}.$$

Note that using the continuous function $\widehat{\varphi} : X_F \to \mathbb{Z}$ defined by $\widehat{\varphi}(\alpha) = \operatorname{sgn}_\alpha \varphi$, each $X_m$ can be expressed as a finite union of clopen sets of the form $\widehat{\varphi}^{-1}(r)$ for some $r \in \mathbb{Z}$, so each $X_m$ is clopen. Only finitely many of them are nonempty. Let $M = \{m_1, \dots, m_r\}$ consist of all the indices $m_i \ge 1$ such that $X_{m_i} \ne \emptyset$. By Lemma 3, for each $m \in M$ there exists $\rho_m \in I^nF$ such that $|\operatorname{sgn}_\alpha(\varphi - \rho_m)| < 2^n$ for $\alpha \in X_m$ and $\operatorname{sgn}_\beta(\varphi - \rho_m) = \operatorname{sgn}_\beta \varphi$ for all $\beta \in X_F \backslash X_m$. Let $\rho = \sum_{i=1}^r \rho_{m_i} \in I^nF$. Since $\operatorname{st}_r F = n - 1$, we can write $\rho = 2\rho_1 + \rho_2$ with $\rho_1 \in I^{n-1}F$ and $\rho_2 \in W_tF$.

Let $\sigma = \varphi - 2\rho_1$. Then

$$|\operatorname{sgn}_\alpha \sigma| = |\operatorname{sgn}_\alpha(\varphi - 2\rho_1)| = |\operatorname{sgn}_\alpha(\varphi - \rho)| \le 2^n - 1$$

for all $\alpha \in X_F$. Thus, $\dim(\sigma_K)_{an} \le u_{2^n-1}(F)$, and as $(2\rho_1)_K = 0$, we get $\dim(\varphi_K)_{an} = \dim(\sigma_K)_{an}$. □

Theorem 2 together with Proposition 1 establishes the following bound for $u(F(\sqrt{-1}))$:

**Corollary 1.** *Let $F$ be a field. If $u(F) < \infty$ and $\operatorname{st}_r F = n < \infty$ then*

$$u(F(\sqrt{-1})) \leq \frac{3}{2}u(F) + 2^{2n} - 2^{n-1}$$

*if $n > 0$ and*

$$u(F(\sqrt{-1})) \leq \frac{3}{2}u(F) + 1$$

*if $n = 0$.*

**Remark 2.** Similarly, one can find bounds for any quadratic extension $K = F(\sqrt{x})$ of a formally real field $F$ with finite reduced stability $\operatorname{st}_r F = n < \infty$. If $x$ is not totally negative over $F$, then the orderings $\alpha \in H(x)$ extend to $K$ (in particular, $K$ will also be formally real). If $\psi \in W_t K$, decompose $\psi$ as in the proof of Theorem 2. The form $\psi_1$ has dimension $\leq \frac{1}{2}u(F)$ and thus $|\operatorname{sgn}_\alpha \varphi| \leq \frac{1}{2}u(F)$ for all $\alpha \in H(x)$. Let the form $\rho_1$ be defined as in the proof but using sets $X_m$ that form a cover of $H(-x)$ instead of $X_F$. Define $\sigma$ as $\varphi - \langle 1, -x\rangle\rho_1$. Its signature $|\operatorname{sgn}_\alpha \sigma|$ is then bounded by $\frac{1}{2}u(F)$ on $H(x)$ and $2^{n+1} - 1$ on $H(-x)$, so that

$$u(F(\sqrt{x})) \leq \frac{3}{2}u(F) + \max\{2^{2n} - 2^{n-1}, 2^{n-2}u(F)\}$$

if $n > 0$, and $u(F(\sqrt{x})) \leq \frac{3}{2}u(F) + \max\{1, \frac{1}{2}u(F)\}$ if $n = 0$.

Combining the bound from Corollary 1 with previously known bounds one obtains bounds for finite extensions $K/F$:

**Corollary 2.** *Let $F$ be a formally real field with $u(F) < \infty$ and $\operatorname{st}_r F = n < \infty$. Let $K/F$ be any finite extension of $F$. Then*

$$u(K) < ([K : F] + 1)(3u(F) + 2^{2n+1} - 2^n)$$

*if $n > 0$ and $u(K) < ([K : F] + 1)(3u(F) + 2)$ if $n = 0$.*

*If $K = K(\sqrt{-1})$ then*

$$u(K) \leq ([K : F] + 1)(\frac{3}{4}u(F) + 2^{2n-1} - 2^{n-2}).$$

*Proof.* $u(K) < 4u(K(\sqrt{-1}))$ by [EL76, Theorem 6.2]. By [Lee84, Theorem 2.10], we have $u(K(\sqrt{-1})) \leq \frac{1}{2}([K : F] + 1)u(F(\sqrt{-1}))$. The estimates then follow from Corollary 1.

□

The following theorem summarizes the behavior of the $u$-invariant and reduced stability under finite field extensions.

**Theorem 3.** *Let $F$ be a field. Then the following are equivalent:*

*(1) $u(F) < \infty$ and $\operatorname{st}_r F < \infty$.*
*(2) $u(F(\sqrt{-1})) < \infty$.*
*(3) $u(K) < \infty$ and $\operatorname{st}_r K < \infty$ for every finite extension $K/F$.*
*(4) There exists a finite extension $E/F$ with $u(E) < \infty$ and $\operatorname{st}_r E < \infty$.*
*(5) $u(K) < \infty$ for every finite extension $K/F$.*

*Proof.* (1)⇒(2): This is Corollary 1.

(2)⇒(1): As $F(\sqrt{-1})$ is non-formally real, $u(F) < \infty$ by [EL76, Theorem G]. If $u(F(\sqrt{-1})) < 2^m$ for some $m \in \mathbb{N}$, then by [EL76, Corollary 4.7(i)], we have $\mathrm{st}_r\, F \leq m$.

(2)⇒(3): Let $K/F$ be any finite extension. Then $u(F(\sqrt{-1})) < \infty$ implies that $u(K(\sqrt{-1})) < \infty$ by [Elm77, Corollary 3.3(ii)], and hence $u(K) < \infty$ by [Elm77, Corollary 3.3(i)]. Again by [EL76, Corollary 4.7(i)], we have $\mathrm{st}_r\, K < \infty$.

(3)⇒(4), (3)⇒(5) and (5)⇒(2) are trivial.

(4)⇒(2): If $E/F$ is a finite extension with $u(E) < \infty$ and $\mathrm{st}_r\, E < \infty$ then $u(E(\sqrt{-1})) < \infty$ by Corollary 1 and hence $u(F(\sqrt{-1})) < \infty$ by [Elm77, Corollary 3.3(i)]. □

## References


[Brö74] Ludwig Bröcker, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math. Ann. **210** (1974), 233–256. MR 0354549 (50 #7027)

[EL72] Richard Elman and T. Y. Lam, *Quadratic forms over formally real fields and pythagorean fields*, Amer. J. Math. **94** (1972), 1155–1194. MR 0314878 (47 #3427)

[EL76] ———, *Quadratic forms under algebraic extensions*, Math. Ann. **219** (1976), no. 1, 21–42. MR 0401649 (53 #5476)

[Elm77] Richard Elman, *Quadratic forms and the u-invariant. III*, Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976), Queen's Univ., Kingston, Ont., 1977, pp. 422–444. Queen's Papers in Pure and Appl. Math., No. 46. MR 0491490 (58 #10732)

[Lee84] David B. Leep, *Systems of quadratic forms*, J. Reine Angew. Math. **350** (1984), 109–116. MR 743536 (85j:11038)

[Mar80a] Murray Marshall, *Spaces of orderings. IV*, Canad. J. Math. **32** (1980), no. 3, 603–627. MR 586979 (81m:10035)

[Mar80b] ———, *The Witt ring of a space of orderings*, Trans. Amer. Math. Soc. **258** (1980), no. 2, 505–521. MR 558187 (81b:10012)

[Mar96] Murray A. Marshall, *Spaces of orderings and abstract real spectra*, Lecture Notes in Mathematics, vol. 1636, Springer-Verlag, Berlin, 1996. MR 1438785 (98b:14041)



Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095-1555

*E-mail address*: schub@ucla.edu