WEAKLY ISOTROPIC QUADRATIC FORMS UNDER FIELD EXTENSIONS

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Abstract. For a field F of characteristic not 2, let $\widehat{u}(F)$ denote the maximal dimension of anisotropic, weakly isotropic, non-degenerate quadratic forms over F. In this paper, we investigate the behavior of this invariant under field extensions.

1. Introduction and Preliminaries

Let F be a field of characteristic not 2. In this paper, a "(quadratic) form" over F always means a non-degenerate quadratic form. Denote by WF the Witt ring of F, i.e., the ring of equivalence classes of non-degenerate quadratic forms over F. We denote the torsion subgroup of WF by W_tF. Over F, we consider three classes of quadratic forms and define

$u(F) := \max\{\dim \varphi_{an} \mid \varphi \in W_tF\}$, the "(general) u-invariant" of F

$\widehat{u}(F) := \max\{\dim \varphi_{an} \mid \varphi \in WF$ weakly isotropic$\}$

$\widetilde{u}(F) := \max\{\dim \varphi_{an} \mid \varphi \in WF$ totally indefinite$\}$, the "Hasse number" of F

Over any field F we have $u(F) \leq \widehat{u}(F) \leq \widetilde{u}(F)$. Over non-formally real fields these invariants all coincide. One is interested in the behavior of these invariants under field extensions, specifically of conditions for when "Going Up" and "Going Down" hold (i.e., finiteness of the respective invariant in the base field implies finiteness in the extension field, and vice versa). Going Up and Going Down of finiteness of the u-invariant is well-understood at least under finite field extensions (see [EL76], [Elm77], [Sch07]). Similarly, Going Up and Going Down of finiteness of the Hasse number under finite field extensions has been settled (see [Pre78], [PW79],[EP84]). The $\widehat{u}$-invariant has not been studied as extensively as the others. If the Hasse number of a field F is finite, it is already equal to $\widehat{u}(F)$. Recently, K. Becher showed that $\widehat{u}(F)$ is a meaningful invariant in its own right by constructing formally real fields F with I^3F torsion-free, $u(F) = 2m$, $\widehat{u}(F) = 2n$ and $\widetilde{u}(F) = \infty$ for any $2 \leq m \leq n < \infty$ ([Bec06b]). In general, this invariant appears more difficult to handle than both the u-invariant and the Hasse number. Nevertheless, a few known results have analogues for the $\widehat{u}$-invariant. In the current paper, we start investigating the behavior of this invariant under field extensions, with a focus on quadratic extensions.

Section 2 uses a construction by A. Prestel (cf. [Pre78]) to show that finiteness of $\widehat{u}(F)$ does not, in general, go down quadratic extensions. A different type of construction in Section 3 will show that even in case of finite reduced stability,

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finiteness of $\widehat{u}(F)$ does not, in general, go up quadratic extensions either. In Section 5 we exploit the connection between the property S_1 and finiteness of $\widehat{u}(F)$ to prove a Going Up result for totally positive quadratic extensions and Going Down for any finite extension K/F such that all the orderings of F extend to K, under the additional assumption that F has finite ordering space.

We use standard notation and results from the algebraic theory of quadratic forms over fields as presented in [Lam05] or [EKM08].

If F is a formally real field, then an *ordering* of F is a subset $\alpha \subset F^\times$ satisfying (1) $\alpha+\alpha \subset \alpha$, (2) $\alpha \cdot \alpha \subset \alpha$, (3) $\alpha \cap -\alpha = \emptyset$ and (4) $\alpha \cup -\alpha = F^\times$. An element $a \in F^\times$ is called *positive with respect to* α if $a \in \alpha$ and *negative with respect to* α otherwise. If $\varphi \cong \langle a_1, \dots, a_n \rangle$ is a form over F and r is the number of a_i that are positive and s the number of a_i that are negative with respect to α, then the *signature of* φ *at* α is $\operatorname{sgn}_\alpha \varphi := r - s$. A form φ is called *definite at* α if $|\operatorname{sgn}_\alpha \varphi| = \dim \varphi$ and *indefinite at* α otherwise. A form is called *totally indefinite* if it is indefinite at every ordering α.

The set of orderings of a formally real field F, denoted by X_F, is a topological space with a subbasis given by the Harrison sets $H(a)$ consisting of all the orderings α such that a is positive with respect to α.

We will use some results from a more abstract version of these concepts but translate these into the field case in this paper. *Spaces of orderings* (X, G) are defined in [Mar96]. In the case of a formally real field F, we denote by G_F the group of *generalized square classes* $F^\times/\sigma(F)$, where $\sigma(F) = \{x_1^2 + \dots + x_n^2 \mid x_i \in F^\times, i = 1, \dots, n\}$. We may view an element of X_F as a character on G_F (i.e., a group homomorphism from G_F to $\{-1, 1\}$) and dually, an element of G_F as a map from X_F to $\{-1, 1\}$. Then (X_F, G_F) is a space of orderings in the sense of [Mar96]. The *chain length* $cl(X_F)$ of X_F is the maximum integer k such that there are $a_0, \dots, a_k \in G_F$ with $H(a_0) \subsetneqq H(a_1) \subsetneqq \dots \subsetneqq H(a_k)$, or ∞ if no such maximum exists. A *morphism of spaces of orderings* is a map $f : X_2 \to X_1$ such that for each $g \in G_1$ the map $g \circ f : X_2 \to \{\pm 1\}$ is an element of G_2. The induced group homomorphism $G_1 \to G_2$ sending g to $g \circ f$ will also be denoted by f. In particular, if X_K and X_F are the ordering spaces of formally real fields K and F, respectively, then this construction leads to a ring homomorphism $W_{red}F \to W_{red}K$, where $W_{red}F := WF/W_tF$.

Given spaces of orderings (X_i, G_i), $i = 1, \dots, n$, their *direct sum* $\bigoplus_{i=1}^n (X_i, G_i)$ is the space (X, G), where X is the disjoint union of the X_i and G is the direct product of the G_i. If all the X_i have finite chain length, then (X, G) is realizable as the ordering space of a formally real Pythagorean field F (cf. [Mar96]).

If G is any multiplicative group of exponent 2 with a distinguished element $e \neq 1$, then a *fan* is the pair (T, G) where $T = \{x \in \operatorname{Hom}\{G, \{\pm 1\}\} \mid x(e) = -1\}$. The *stability* $\operatorname{st} X$ of a space of orderings (X, G) is the maximum integer n such that there exists a fan (T, H) with $T \subset X$, $H \subset G$ and $|T| = 2^n$, or ∞ if no such maximum exists. If (X_F, G_F) is the space of orderings of a formally real field F, then the stability of X_F is called the *reduced stability of* F, denoted by $\operatorname{st}_r F$ (this definition of reduced stability is equivalent to the "usual" definition, cf. [Mar96]). We say F satisfies the *strong approximation property (SAP)* if $\operatorname{st}_r F \leq 1$. By [Brö74, Satz 3.17], $\operatorname{st}_r F \leq n$ if and only if $I^{n+1}F = 2I^nF + I_t^{n+1}F$ where I^nF is the n-th power of the fundamental ideal IF in the Witt ring WF and $I_t^nF = I^nF \cap W_tF$.

For a valuation ν on F we will denote by Γ_ν and $\overline{F}_\nu$ the value group and residue class field of ν, respectively.

A quadratic form φ is called *weakly isotropic* if $m\varphi$ is isotropic for some integer m. Equivalently, $\varphi \cong \langle a_1, \dots, a_n\rangle$ is weakly isotropic if and only if there are $w_i \in \sigma(F)$ such that $\sum_{i=1}^n a_i w_i = 0$ ([Pre84, Proposition 2.7]). A form that is not weakly isotropic is called *strongly anisotropic*. Clearly, every form that is definite at some ordering is strongly anisotropic, so every weakly isotropic form is totally indefinite. The converse is false in general.

Fact 1.1. *Let F be a formally real field.*

1. *If $\widetilde{u}(F)$ is finite, then F satisfies SAP.*
2. *F satisfies SAP if and only if every totally indefinite form is weakly isotropic. Hence $\widetilde{u}(F) = \widehat{u}(F)$ for a SAP-field F.*
3. *If F is Pythagorean, then $u(F) = \widehat{u}(F) = 0$.*

Proof. (1) This is [ELP73, Theorem B].

(2) This follows from [ELP73, Theorem C] together with [Brö74, Folgerung 2.12].

(3) This is immediately clear from the equivalent definition of “weakly isotropic”, since over a Pythagorean field $\sigma(F) = (F^\times)^2$. □

If F is formally real, then $u(F)$ is always even (if it is finite), since every torsion form has to have signature 0 and hence even dimension. If, in addition, I^3F is torsion-free, then $\widetilde{u}(F)$ is even as well (if it is finite); cf. [ELP73, Theorem H]. In [Bec06b, Remark 5.5(2)] the question was raised if $\widehat{u}(F)$ will always be even, at least assuming I^3F is torsion-free. Changing “totally indefinite” to “weakly isotropic” in the proof of [ELP73, Theorem H] shows this to be true; the proof works without further modifications. Thus, we have:

Proposition 1.2. *Let F be a formally real field satisfying I^3F is torsion-free. If $\widehat{u}(F)$ is finite, then $\widehat{u}(F)$ is even.*

Thus, all the possible values for $\widehat{u}(F)$ are known when I^3F is torsion-free (cf. [Hof01] in the case $\widehat{u}(F) = \widetilde{u}(F) < \infty$ and [Bec06b] in the case $\widehat{u}(F) < \widetilde{u}(F) = \infty$).

We begin with some basic observations.

Proposition 1.3.
1. *If $[K : F]$ is finite and odd or K/F is purely transcendental then $\widehat{u}(F) \le \widehat{u}(K)$.*
2. *If F is formally real, then $\widehat{u}(F((t))) < \infty$ if and only if F is Pythagorean. In particular, $\widehat{u}(F((t))) < \infty$ if and only if $\widehat{u}(F((t))) = 0$.*
3. *There exists a field F and a formally real extension K/F with $[K : F] = 3$ such that $\widehat{u}(F) < \infty$ but $\widehat{u}(K) = \infty$.*

Proof. (1) This follows from Springer's Theorem.

(2) If F is Pythagorean, then so is $F((t))$. Thus $\widehat{u}(F((t))) = 0$ by Fact 1.1. Conversely, if F is not Pythagorean then there exists $w \in \sigma(F)$ that is not a square. Then the form $\varphi_n = n\langle 1\rangle \perp \langle t, -wt\rangle$ is anisotropic and weakly isotropic over $F((t))$ for every n, so $\widehat{u}(F((t))) = \infty$.

(3) Let F_0 be any formally real, Pythagorean field with a field extension K_0/F_0 of degree 3 such that K_0 is not Pythagorean. Let $F = F_0((t))$ and $K = K_0((t))$. Then $\widehat{u}(F) = 0$ and $\widehat{u}(K) = \infty$ by (2). □

Statements (1) and (3) in the Proposition hold with $\widetilde{u}(F)$ in place of $\widehat{u}(F)$. The corresponding statement to (2) is: if F is formally real, then $\widetilde{u}(F((t))) < \infty$ if and only if F is Euclidean, i.e., F is Pythagorean and has a unique ordering (cf. [ELP73, Proposition 1]). The additional conclusion that F has a unique ordering, however, is false in the case of $\widehat{u}(F((t)))$. In fact, any ordering space (X, G) of finite chain length is realizable as the ordering space X_F of some Pythagorean field F, and thus $\widehat{u}(F((t))) = 0$ even though $|X_F|$ may equal any $n \in \mathbb{N}$ or ∞.

2. Going Down under Quadratic Extensions

A. Prestel showed that for any given $n \geq 2$ there is a formally real field F and formally real extension $F(\sqrt{a})$ such that $\widetilde{u}(F(\sqrt{a})) = 2^n$ and $\widetilde{u}(F) = \infty$ ([Pre78, Theorem 3.3]). We show in this section that this construction yields the analogous statements for the $\widehat{u}(F)$-invariant.

Theorem 2.1. *For every $2 \leq n < \infty$ there is a field F and a formally real extension $F(\sqrt{a})$ with $\widehat{u}(F) = \infty$ and $\widehat{u}(F(\sqrt{a})) = 2^n$.*

Proof. The construction from the proof of [Pre78, Theorem 3.3] can be used without modifications; one only has to show that $\widehat{u}(F) = \infty$ still.

Let $\widetilde{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$ algebraically independent over $\widetilde{\mathbb{Q}}$. Let $K = \widetilde{\mathbb{Q}}(x_1, \ldots, x_n)$ and let $x \in \mathbb{R}$ be positive and transcendental over K. Let $K_1 = K(x)(\sqrt[2^m]{x} \mid m \in \mathbb{N})$ with valuations v_1 induced by the ordering of $\mathbb{R}$ and v_2 the unique extension of the place induced by $x = 0$ from $K(x)$. Let H_i be the Henselizations of K_1 with respect to v_i and $F = H_1 \cap H_2$. Consider the forms $\varphi_n = n\langle 1\rangle \perp \langle x_1, -x_1(1 + x_2^2)\rangle$ with $n \in \mathbb{N}$, which are weakly isotropic over F. If φ_n is isotropic for some n, then $(\varphi_n)_{H_2}$ is isotropic as well and thus (using the place induced by $x = 0$), $(\varphi_n)_K$ is isotropic, a contradiction. Thus $\widehat{u}(F) = \infty$. Now let $a = m^2x - 1 \in F^\times$ for some m satisfying $a > 0$ in $\mathbb{R}$. Then following the proof of [Pre78, Theorem 3.3], we have $\widetilde{u}(F(\sqrt{a})) = 2^n$, and thus $\widehat{u}(F(\sqrt{a})) = 2^n$ by Fact 1.1. □

Remark 2.2. Just as in the case of the u-invariant and Hasse number, $\widehat{u}(F) \leq 2$ will go down any quadratic extension: for any formally real field F we have $u(F) \leq 2 \Rightarrow \widehat{u}(F) = u(F)$ ([Bec06b, Proposition 5.3]). Thus if $\widehat{u}(F(\sqrt{a})) \leq 2$ then $u(F(\sqrt{a})) \leq 2$, so by [EL76, Corollary 4.13], $u(F) \leq 2$ and hence $\widehat{u}(F) \leq 2$.

3. Reduced Stability and Going Up under Quadratic Extensions

L. Bröcker showed that if K/F is algebraic, then $\mathrm{st}_r K \leq \mathrm{st}_r F + 1$ [Brö74, Satz 4.3] and if K/F is purely transcendental of transcendence degree n then $\mathrm{st}_r(K) \in \{\mathrm{st}_r F + n, \mathrm{st}_r F + n + 1\}$ [Brö74, Satz 4.9]. R. Bos showed that if K/F is any finite field extension such that all orderings of F extend to K, then $\mathrm{st}_r F \leq \mathrm{st}_r K$ (Corollary 6.13 in [Bos84], which unfortunately remains unpublished and is not widely available). A different proof, due to K. Becher will appear in a forthcoming article. The *absolute stability* $\mathrm{st}_a F$ of F is defined to be the minimum $n \in \mathbb{N}$ such that $I^{n+1}F = 2I^nF$ (or ∞ if no such n exists). Clearly, $\mathrm{st}_r F \leq \mathrm{st}_a F$ and we have equality if F is Pythagorean. By [EL76, Corollaries 4.6 and 4.7], $\mathrm{st}_a F$ increases by at most 1 (and does not decrease) when going up a quadratic extension.

Note that if $\mathrm{st}_r F \leq 1$ then F satisfies SAP, so $\widehat{u}(F) = \widetilde{u}(F)$ by Fact 1.1. Then finiteness of $\widetilde{u}(F)$ (and hence $\widehat{u}(F)$) goes up any quadratic extension by [EP84, Corollary 2.6]. Also, the u-invariant goes up a quadratic extension whenever the base field has finite u-invariant and reduced stability ([Sch07, Theorem 3]). The following theorem shows that the $\widehat{u}$-invariant may go from finite to infinite when going up a quadratic extension even if the base field has finite reduced stability. Thus, in a sense the $\widehat{u}$-invariant is the least "well-behaved" invariant of the three.

Theorem 3.1. *For every $1 < n \leq \infty$ there exists a formally real field F and a formally real quadratic extension K/F such that $\mathrm{st}_r F = n$, $u(F) = \widehat{u}(F) = 0$, $u(K) < \infty$ (if $n < \infty$), and $\widehat{u}(K) = \infty$.*

Proof. Let (X_1, G_1) be the space of orderings of $\mathbb{R}$, and (X_2, G_2) the space of orderings of $\mathbb{R}((t_1))((t_2))\ldots((t_{n-1}))$ if $n \in \mathbb{N}$, or $\mathbb{R}((t_1))((t_2))\ldots$ if $n = \infty$, so that $|X_1| = 1$ and X_2 is a fan of size $|X_2| = 2^{n-1}$ (or $|X_2| = \infty$ if $n = \infty$). Let (X, G) denote the direct sum $(X_1, G_1) \oplus (X_2, G_2)$. We will identify G with $G_1 \times G_2$. By [Mar96, Theorem 4.1.1(1)], (X, G) is a space of orderings. Let $g = (1_{G_1}, -1_{G_2}) \in G$. Then $X_1 = H(g)$. Note that by [Mar96, Theorem 4.2.1], the chain length of X is $cl(X) = cl(X_1) + cl(X_2) \leq 3$. It follows (cf. [Mar96, Chap. 4.2]) that (X, G) is realizable as the space of orderings of a Pythagorean field F_0 with $G = F_0^\times/\sigma(F_0) = F_0^\times/(F_0^\times)^2$. As fans are completely contained in connected components of (X, G), we have $\mathrm{st}_r F_0 = n - 1$.

Pick a representative $a \in F_0^\times$ of the square class $g \in G$, so that $|H(a)| = 1$. The only ordering of F_0 that extends to $K_0 = F_0(\sqrt{a})$ is the unique $\alpha \in H(a)$, and there are two corresponding orderings in X_{K_0}, one making $\sqrt{a}$ positive and one making $\sqrt{a}$ negative. Thus, $|X_{K_0}| = 2$ and $|G_{K_0}| = 4$ and hence $\mathrm{st}_r K_0 = 1$. In particular, K_0 is formally real and not Pythagorean.

Let $F = F_0((t))$ and $K = K_0((t))$. Then F is Pythagorean, so $\widehat{u}(F) = 0$, and $\mathrm{st}_r F = \mathrm{st}_r F_0 + 1 = n$. Pick any totally positive element w in K_0 that is not a square. Then $\varphi = m\langle 1\rangle \perp t\langle 1, -w\rangle$ is anisotropic and weakly isotropic over K for any m which shows that $\widehat{u}(K) = \infty$. Finally, if $n < \infty$ then $u(K_0) < \infty$ by [Sch07, Theorem 3], and thus $u(K) = 2u(K_0) < \infty$. □

Remark 3.2. The proof of Theorem 3.1 can be modified to construct a formally real quadratic extension K/F satisfying $\mathrm{st}_r F = n$ and $\mathrm{st}_r K = m$ for any $1 \leq m \leq n \leq \infty$. A more precise version of this result will appear in a forthcoming article.

[EP84, Theorem 5.1] gives a counterexample to finiteness of $\widehat{u}(F)$ going up a formally real quadratic extension with the base field satisfying $\widehat{u}(F) \neq 0$. In this example, however, we have $\mathrm{st}_r F = \infty$.

Proposition 3.3. *There exists a formally real field F and a formally real quadratic extension $F(\sqrt{a})$ such that $\widehat{u}(F) = 2$ and $\widehat{u}(F(\sqrt{a})) = \infty$.*

Proof. By [EP84, Theorem 5.1], there exists a formally real field F and a formally real quadratic extension $F(\sqrt{a})$ with $u(F) = 2$ and $u(F(\sqrt{a})) = \infty$. Thus $\widehat{u}(F) = 2$ by [Bec06b, Proposition 5.3], and clearly $\widehat{u}(F(\sqrt{a})) = \infty$. □

4. The $\widehat{u}$-Invariant and the Property S_1

We recall the following definitions (cf. [PW79]): First, a form φ is said to be *almost isotropic* if for every diagonalization $\varphi \cong \langle a_1, \ldots, a_n\rangle$ there exists some $m \in \mathbb{N}$ such that $m\langle a_1\rangle \perp \langle a_2, \ldots, a_n\rangle$ is isotropic. Second, a formally real field F is said to satisfy the property S_1 if any quadratic form $\langle 1, -w\rangle$ with $w \in \sigma(F)$ represents every coset of $F^\times$ modulo $\sigma(F)$.

In this section, we will use the relationship

$$\text{almost isotropic} \underset{\text{iff } S_1}{\overset{\Rightarrow}{\Leftarrow}} \text{weakly isotropic} \underset{\text{iff SAP}}{\overset{\Rightarrow}{\Leftarrow}} \text{totally indefinite}$$

to prove some statements about $\widehat{u}(F)$ under the assumption that F satisfies S_1 that are analogous to statements about $\widetilde{u}(F)$ under the assumption that F satisfies SAP and S_1. The implications from left to right are clear. The statement "SAP if and only if (totally indefinite $\Rightarrow$ weakly isotropic)" is Fact 1.1(2). Thus, there is only one statement to prove:

Proposition 4.1. *Let F be a formally real field. Then F satisfies S_1 if and only if every weakly isotropic form over F is almost isotropic.*

Proof. The proof of "only if" is implicit in the proof of (3)$\Rightarrow$(4) in [PW79, Theorem 2]. Conversely, assume that every weakly isotropic form over F is almost isotropic. Let $w \in \sigma(F)$ and $a \in F^\times$. We wish to show that $\langle 1, -w\rangle$ represents some coset of a modulo $\sigma(F)$. Now $\langle 1, -w\rangle$ is torsion, hence the form $\varphi \cong \langle 1, -w, -a\rangle$ is weakly isotropic. By assumption, φ is almost isotropic, so there exists some m such that $\langle 1, -w\rangle \perp m\langle -a\rangle$ is isotropic, say $x_1^2 - wx_2^2 - a\sum_{i=1}^m y_i^2 = 0$. Let $w_1 = \sum_{i=1}^m y_i^2 \in \sigma(F)$. Then $\langle 1, -w, -aw_1\rangle$ is isotropic and hence $\langle 1, -w\rangle$ represents aw_1. □

Remark 4.2. This proposition should be compared with the analogous statement [PW79, Theorem 1] which says that a formally real field F satisfies S_1 and SAP if and only if every totally indefinite form over F is almost isotropic.

Corollary 4.3. *If F is formally real and $\widehat{u}(F) < \infty$, then F satisfies S_1.*

Proof. By Proposition 4.1, it suffices to show that every weakly isotropic form is almost isotropic. Let $\varphi \cong \langle a_1, \ldots, a_n\rangle$ be weakly isotropic. Then certainly for any $m \in \mathbb{N}$, the form $m\langle a_1\rangle \perp \langle a_2, \ldots, a_n\rangle$ is weakly isotropic as well. In particular, picking $m \geq \widehat{u}(F) - n + 2$, this form has to be isotropic. Thus φ is almost isotropic. □

Corollary 4.4.
(1) *If K/F is a finitely generated, formally real extension of transcendence degree ≥ 2, then $\widehat{u}(K) = \infty$.*
(2) *There exists a hereditarily Pythagorean field F with $\widehat{u}(F(t)) = \infty$.*

Proof. (1) By [EP84, Lemma 4.4], no such field K will satisfy S_1. Thus $\widehat{u}(K) = \infty$ by Corollary 4.3.

(2) By [Mei82, Satz 6] there exists a hereditarily Pythagorean field F such that $F(t)$ does not satisfy S_1. Thus $\widehat{u}(F(t)) = \infty$ by Corollary 4.3. □

As another application of Corollary 4.3, we show that every formally real field F with $\widehat{u}(F) < \infty$ satisfies the property WD ("weak decomposition property") introduced by K. Becher in [Bec06a]: F is a WD-field if every form φ over F can be decomposed as $\varphi \cong \psi \perp \beta_1 \perp \ldots \perp \beta_r$ where ψ is strongly anisotropic and β_i are binary torsion forms (possibly $r = 0$).

Proposition 4.5. *If F is a formally real field satisfying S_1, then every weakly isotropic form φ contains a binary torsion subform. (In particular, φ represents every coset of $F^\times/\sigma(F)$.)*

Proof. Let φ be a weakly isotropic form over F, say $\varphi \cong \langle a_1, \dots, a_n\rangle$. By Proposition 4.1, φ is almost isotropic. Hence there exists an $m \in \mathbb{N}$ such that $m\langle a_1\rangle \perp \langle a_2, \dots, a_n\rangle$ is isotropic. In particular, there exists a totally positive element $w \in \sigma(F)$ such that $\langle wa_1, a_2, \dots, a_n\rangle$ is isotropic. Thus, $\langle a_2, \dots, a_n\rangle$ represents $-wa_1$, say $\langle a_2, \dots, a_n\rangle \cong \langle -wa_1\rangle \perp \varphi_1$. Then $\varphi \cong \langle a_1, -wa_1\rangle \perp \varphi_1$ as needed. □

Proposition 4.5 and Corollary 4.3 immediately give

Corollary 4.6. *Let F be a formally real field. If F satisfies S_1, then F is a WD-field. In particular, this is the case if $\widehat{u}(F) < \infty$.*

The converse of Corollary 4.6 is false:

Example 4.7. Let F be a hereditarily Pythagorean number field that is not Euclidean. By [Mei82, Satz 6], $F(t)$ does not satisfy S_1, but it satisfies WD, cf. [Bec06a, Example 4.3(4)].

We have the following weak converse of Corollary 4.3.

Proposition 4.8. *Let F be a formally real field with $u(F(\sqrt{-1})) < \infty$ and satisfying S_1. Then $\widehat{u}(F) < \infty$.*

Proof. Assume $u(F(\sqrt{-1})) < \infty$. It follows that $u(F) < \infty$ (see for example [EKM08, Theorem 37.4]), and hence the Pythagoras number $p(F) < \infty$. In complete analogue to the proof of [EP84, Theorem 2.5], if φ is a weakly isotropic form over F of large enough dimension, then $\varphi \cong \psi \perp 2^{p(F)}\langle x\rangle$ for some $x \in F^\times$. The form $\psi \perp \langle x\rangle$ is then weakly isotropic as well, so it is almost isotropic by Proposition 4.1. Thus, φ is isotropic. □

Corollary 4.9. *Let F be a formally real field with finite reduced stability. Then*

$$u(F) < \infty \text{ and } F \text{ satisfies } S_1 \Leftrightarrow \widehat{u}(F) < \infty.$$

Proof. "$\Leftarrow$": F satisfies S_1 by Corollary 4.3, and $u(F) \le \widehat{u}(F) < \infty$.

"$\Rightarrow$": By Proposition 4.8, it suffices to show that $u(F(\sqrt{-1})) < \infty$. Since $\mathrm{st}_r F < \infty$ and $u(F) < \infty$, this follows from [Sch07, Corollary 1]. □

Remark 4.10. If in Corollary 4.9 we have $\mathrm{st}_r F \le 1$, then in fact $\widetilde{u}(F)$ is finite by [EP84, Theorem 2.5]. Corollary 4.9 is an appropriate generalization of this. In the case of the Hasse number, finiteness of $\widetilde{u}(F)$ already implies $\mathrm{st}_r F \le 1$, but finiteness of $\widehat{u}(F)$ does not imply $\mathrm{st}_r F < \infty$ as Example 4.11 shows.

Example 4.11. Let $F = \mathbb{R}((t_1))((t_2))((t_3))\dots$, the field of iterated Laurent series in infinitely many variables over $\mathbb{R}$. Then $\widehat{u}(F) = 0$ as F is Pythagorean (cf. Fact 1.1), but $\mathrm{st}_r F = \infty$. In particular, the converse of Proposition 4.8 is false, since $u(F(\sqrt{-1})) = \infty$.

Theorem 3.1 shows that finite u-invariant and finite reduced stability together are not sufficient for finite $\widehat{u}$-invariant. Theorem 4.12 below shows that the condition S_1 together with finite u-invariant are not sufficient either. The proof was shown to me by K. Becher. Note that the *Pythagorean index pind*(A) of a central simple algebra A over a field F is defined in [Bec06b] as the (Schur) index of A over the Pythagorean closure of F.

Theorem 4.12. *There exists a formally real field F with $u(F) = 4$, satisfying S_1, and $\widehat{u}(F) = \infty$.*

Proof. Let F_1 be a formally real field and Q_1 a quaternion division algebra over F_1 whose norm form is torsion, e.g., $F_1 = \mathbb{Q}$ and $Q_1 = (-1, 7)$. Now suppose we have a formally real field F_n with a division algebra $A_n = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ such that $pind(A_n) = 2^{n-1}$. Let $F'_{n+1} = F_n(x_n, y_n)$ where x_n and y_n are indeterminates over F_n, and let Q_{n+1} be the quaternion algebra (x_n, y_n). Then $A_{n+1} = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_{n+1}$ is a division algebra over F'_{n+1} with $pind(A_{n+1}) = 2^n$. Now let F_{n+1} be the compositium of all function fields $F'_{n+1}(\varphi)$ where φ is a weakly isotropic form of dimension $> 2n+2$ or a torsion form of dimension > 4 over F'_{n+1}. Just as in the proof of [Bec06b, Theorem 5.4], it follows that A_{n+1} is a division algebra with $pind(A_{n+1}) = 2^n$ over F_{n+1}. Furthermore, we have $u(F_{n+1}) \leq 4$ and $\widehat{u}(F_{n+1}) \leq 2n + 2$.

After having thus defined recursively a tower of fields $(F_n)_{n\in\mathbb{N}}$, let $F = \cup_{n\geq 1} F_n$. From the above construction, we have a family of quaternion algebras $\{Q_i : i \in \mathbb{N}\}$ over F such that, for every $n \in \mathbb{N}$, $A_n = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_n$ has $ind(A_n) = 2^n$ and $pind(A_n) = 2^{n-1}$. We define forms ψ_n inductively: let $\psi_1 = \langle 1, -x_1, -1-y_1^2, x_1(1+y_1^2)\rangle$ and if $\psi_n = \langle 1\rangle \perp \psi'_n$, let $\psi_{n+1} = \langle 1, -x_{n+1}, -y_{n+1}\rangle \perp -x_{n+1}y_{n+1}\psi'_n$. Note that the index of A_n equals 2^n, $\dim \psi_n = 2n + 2$, and the Clifford algebra of ψ_n is $M_2(A_n)$. Thus, each ψ_n is anisotropic. Also note that each of these forms contains a form similar to ψ'_1 and hence is weakly isotropic. Thus, $\widehat{u}(F) = \infty$. Since $u(F) \leq 4$, we have $u(F) = 4$ by [Bec06b, Proposition 5.3]. Finally, each F_n satisfies S_1 by Corollary 4.3, so F does as well. □

5. Finite Field Extensions that Preserve Orderings

We show that at least in case F only admits finitely many orderings, the property S_1 will go up totally positive quadratic extensions and down any finite extension K/F such that all orderings of F extend to K. This will lead to Going Up/Going Down-results in those cases.

Proposition 5.1. *Let F be a formally real field satisfying S_1, and assume $|X_F| < \infty$. Let $w \in \sigma(F)$. Then $F(\sqrt{w})$ satisfies S_1.*

Proof. We may assume w is not a square in F. First note that $|X_{F(\sqrt{w})}| = 2|X_F|$ is finite as well. [Mei82, Satz 7] shows that a formally real field K with finitely many orderings satisfies S_1 if and only if for all real valuations $v : K^\times \to \Gamma_v$ with $\Gamma_v \neq 2\Gamma_v$, the residue class field $\overline{K}_v$ is Pythagorean.

Assume that F satisfies S_1. Let $v' : F(\sqrt{w})^\times \to \Gamma_{v'}$ be a real valuation with $\Gamma_{v'} \neq 2\Gamma_{v'}$ and denote by v its restriction to F. Then $\Gamma_v \neq 2\Gamma_v$ as well and hence $\overline{F}_v$ is Pythagorean by assumption. Since $\overline{F(\sqrt{w})}_{v'}/\overline{F}_v$ is a totally positive extension, $\overline{F(\sqrt{w})}_{v'} = \overline{F}_v$ is Pythagorean as well. □

Corollary 5.2. *Let F be a formally real field satisfying $|X_F| < \infty$ and $\widehat{u}(F) < \infty$. Let $w \in \sigma(F)$. Then $\widehat{u}(F(\sqrt{w})) < \infty$.*

Proof. Assume that $\widehat{u}(F)$ is finite. By Corollary 4.3 and Proposition 5.1, this implies that both F and $F(\sqrt{w})$ satisfy S_1. Also, finiteness of the u-invariant is preserved when going up a totally positive extension ([EL76, Theorem 7.4]). Finally, since every formally real field with finitely many orderings certainly has finite reduced stability, Corollary 4.9 implies that $\widehat{u}(F(\sqrt{w}))$ is finite. □

Proposition 5.3. *Let F be a formally real field satisfying $|X_F| < \infty$ and K/F a finite extension such that all orderings of F extend to K. Assume K satisfies S_1. Then F satisfies S_1.*

Proof. We will again use the characterization of S_1 used in the proof of Proposition 5.1. Note that since K/F is finite, each ordering of F extends to K in finitely many ways; thus, $|X_K| < \infty$ as well.

Assume that K satisfies S_1. Let $v : F^\times \to \Gamma_v$ be a real valuation of F with $\Gamma_v \neq 2\Gamma_v$. Then v extends to a real valuation $v' : K^\times \to \Gamma_{v'}$ since all orderings of F extend to K. Then also $\Gamma_{v'} \neq 2\Gamma_{v'}$ (as $|\Gamma_{v'}/2\Gamma_{v'}| = |\Gamma_v/2\Gamma_v|$) and by assumption, $\overline{K}_{v'}$ is Pythagorean. Since $[\overline{K}_{v'} : \overline{F}_v] < \infty$, this implies $\overline{F}_v$ is Pythagorean as well, and so F satisfies S_1. □

Corollary 5.4. *Let F be a formally real field satisfying $|X_F| < \infty$. Let K be a finite extension of F such that all orderings of F extend to K. If $\widehat{u}(K) < \infty$, then $\widehat{u}(F) < \infty$.*

Proof. Assume that $\widehat{u}(K)$ is finite. By Corollary 4.3 and Proposition 5.3, this implies that K and hence F satisfy S_1. Also, since $u(K) < \infty$ and K has finite reduced stability, we have $u(F) < \infty$ by [Sch07, Theorem 3]. Again, by Corollary 4.9, $\widehat{u}(F)$ is finite. □

Remark 5.5. Even though it is reasonable to conjecture that the results in this section hold without the restriction of $|X_F|$ being finite, they cannot be proved in this more general case using the above method. While the residue class fields in the above proofs will still be Pythagorean, this is no longer enough to imply S_1 if the field in question has infinitely many orderings.

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References

[Bec06a] Karim Johannes Becher, *Minimal weakly isotropic forms*, Math. Z. **252** (2006), no. 1, 91–102. MR 2209153 (2006k:11060)

[Bec06b] ———, *Totally positive extensions and weakly isotropic forms*, Manuscripta Math. **120** (2006), no. 1, 83–90. MR 2223482

[Bos84] Rikkert Bos, *Quadratic forms, orderings and abstract Witt rings*, Rijksuniversiteit te Utrecht, Utrecht, 1984, Dissertation, Rijksuniversiteit te Utrecht, Utrecht, 1984, With a Dutch summary. MR 756809 (86h:11033)

[Brö74] Ludwig Bröcker, *Zur Theorie der quadratischen Formen über formal reellen Körpern*, Math. Ann. **210** (1974), 233–256. MR 0354549 (50 #7027)

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008. MR 2427530 (2009d:11062)

[EL76] Richard Elman and T. Y. Lam, *Quadratic forms under algebraic extensions*, Math. Ann. **219** (1976), no. 1, 21–42. MR 0401649 (53 #5476)

[Elm77] Richard Elman, *Quadratic forms and the u-invariant. III*, Conference on Quadratic Forms—1976 (Proc. Conf., Queen's Univ., Kingston, Ont., 1976), Queen's Univ., Kingston, Ont., 1977, pp. 422–444. Queen's Papers in Pure and Appl. Math., No. 46. MR 0491490 (58 #10732)

[ELP73] Richard Elman, Tsit Yuen Lam, and Alexander Prestel, *On some Hasse principles over formally real fields*, Math. Z. **134** (1973), 291–301. MR 0330045 (48 #8384)

[EP84] Richard Elman and Alexander Prestel, *Reduced stability of the Witt ring of a field and its Pythagorean closure*, Amer. J. Math. **106** (1984), no. 5, 1237–1260. MR 761585 (86f:11030)

[Hof01] Detlev W. Hoffmann, *Dimensions of anisotropic indefinite quadratic forms. I*, Proceedings of the Conference on Quadratic Forms and Related Topics (Baton Rouge, LA, 2001), no. Extra Vol., 2001, pp. 183–200 (electronic). MR 1869394 (2002j:11032)

[Lam05] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005. MR 2104929 (2005h:11075)

[Mar96] Murray A. Marshall, *Spaces of orderings and abstract real spectra*, Lecture Notes in Mathematics, vol. 1636, Springer-Verlag, Berlin, 1996. MR 1438785 (98b:14041)

[Mei82] Wilfried Meißner, *Quadratische Räume unendlicher Dimension und partielle Isomorphismen*, Math. Z. **181** (1982), no. 4, 481–495. MR 682669 (84i:10020)

[Pre78] Alexander Prestel, *Remarks on the Pythagoras and Hasse number of real fields*, J. Reine Angew. Math. **303/304** (1978), 284–294. MR 514686 (80d:12020)

[Pre84] ———, *Lectures on formally real fields*, Lecture Notes in Mathematics, vol. 1093, Springer-Verlag, Berlin, 1984. MR 769847 (86h:12013)

[PW79] Alexander Prestel and Roger Ware, *Almost isotropic quadratic forms*, J. London Math. Soc. (2) **19** (1979), no. 2, 241–244. MR 533322 (80e:12024)

[Sch07] Claus Schubert, *Going up of the u-invariant over formally real fields*, Math. Ann. **338** (2007), no. 3, 757–765. MR 2317936 (2008d:11031)

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