An Investigation of Tenth Grade Students' Views of the Purposes of Geometric Proof

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Abstract

This paper describes an investigation of tenth grade students' views of the purposes of geometric proof within the context of their learning. Using de Villiers' (1999) purposes of mathematical proof, students' views of proof were diverse with respect to verification, explanation, communication, systematization, and discovery. Students' views were mainly limited to verification and explanation when discussing two-column geometric proof.

Introduction

In *Principles and Standards for School Mathematics,* the National Council of Teachers of Mathematics [NCTM] (2000) distinguished the understanding of, and the ability to write mathematical proof as an important topic and a desirable goal. NCTM proposed that certain topics, such as mathematical proof, are important in the curriculum when the topic has the "utility to develop other mathematical ideas, in linking different areas of mathematics, or in deepening students' appreciation of mathematics as a discipline and a human creation" (p. 15). NCTM argued that reasoning and proof, which rely heavily on making connections between established ideas, are helpful in understanding newly developed concepts as well as in the discovery of new ideas.

In an epistemological sense, proving a mathematical conjecture is one of several processes involved in developing and justifying new knowledge within the discipline of mathematics (Schwab, 1978). Unlike the natural sciences, mathematics is based on non-empirical objects. While empirical methods such as measuring and experimentation may be used in the proving process to achieve some level of conviction in mathematics, the validity of mathematical conjectures is not based solely on observation. The validity of mathematical conjectures is based on mathematical proof.

Aspects of Understanding Mathematical Proof

One way to explore student's understanding of mathematical proof is through Skemp's (1979) conceptualization for understanding mathematics: instrumental, relational, and formal. *Instrumental understanding* refers to the kind of understanding achieved by students who are able to produce correct answers or procedures in a mathematical problem without necessarily understanding what has been achieved. Students with only an instrumental understanding of proof may resort to the memorization of proofs or the memorization of a sequence of steps in proof writing. Harel and Sowder (1998) believed that instruction emphasizing the structure and symbols of proofs might reinforce students' desires to memorize proofs, which may, in turn, lead to ritualism. Students who engage in ritualism may determine that a proof is valid based solely on its appearance rather than for its content. Empirical studies that have investigated high school students' instrumental understanding of mathematical proof have documented that students generally have difficulty producing algebraic and geometric proofs (Healy & Hoyles, 2000; Kahan, 1999; Senk, 1989)

Relational understanding refers to the kind of understanding achieved when students are able to conceptualize the process or procedures of an algorithm or mathematical process. Students who have achieved a relational understanding of proof understand that once a proof has been given for a generalization, no further justification of a particular case within the domain of the generalization is needed. Students who do not possess a relational understanding of proof often produce inductive arguments in the form of empirical checks even after a proof of a generalization has been given. The results of three other studies (Fischbein & Kedem, 1982; Porteous, 1991; Williams, 1979) also indicated high school students did not fully grasp the power of proven algebraic and geometric generalizations. In a recent study, Healy and Hoyles (2000) found that nearly 40% of high school students in England and Wales, when given a choice to check a particular case of a proven algebraic generalization or to deduce it, chose to empirically check the particular case.

Formal understanding refers to the kind of understanding achieved by students who acknowledge the symbols and methods needed to communicate their ideas within a mathematical community. Students who have a formal understanding of mathematics understand the role of mathematical proof in a public forum as a method for convincing others of the validity of conjectures and to explain and communicate to others why a conjecture is true. In the research literature on mathematical proof, there are several aspects of proof that may be considered formal understanding of proof and have been referred to as the functions (or purposes) of proof (Bell, 1976; de Villiers, 1999; Hanna & Jehnke, 1993; Schoenfeld, 1992). Hanna (2000) has provided a comprehensive list of the various purposes of mathematical proof:

1. verification (concerned with the truth of a statement);
2. explanation (providing insight into why it is true);
3. systematization (the organization of various results into a deductive system of axioms, major concepts and theorems);
4. discovery (the discovery or invention of new results);
5. communication (the transmission of mathematical knowledge);
6. construction of an empirical theory;
7. exploration of the meaning of a definition or the consequences of an assumption;
8. incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective. (p. 9).

Researchers have also generally characterized high school students as empiricists, that is, relying mainly on the demonstration of specific examples to establish the validity of a mathematical conjecture. This finding has been evidenced in several studies where students were asked to explain, justify, or prove a mathematical conjecture (Balacheff, 1988; Bell, 1976; Galbraith, 1981; Schoenfeld, 1983) and to convince others about the truth of a mathematical conjecture (Williams, 1979). In a study regarding students' views of the purposes of algebraic proof, Healy and Hoyles (2000) asked students to choose the argument that their classroom teacher would give the highest grade. This line of questioning enabled the researchers to get a better notion of whether students valued deductive proofs, even if they were not able to write proofs. Chazen (1993) examined the explanatory and verification features of geometric proof within the context of an experimental classroom using dynamic geometry software with high school students. While Chazen's examination of high school students' views of verification and explanatory features of geometric proof is important, examining students' views of other purposes of proof is also needed to provide a more holistic description.

Purpose of the Study

This study aimed to describe the breadth and depth of high school students' views of the purposes of geometric proof in the context of learning geometric proof. Thus, the primary purpose of this study was guided by the main research question: What are the views of tenth graders regarding the purposes of geometric proof in the context of learning geometric proof? In addition to providing descriptions of students' formal understanding, this study provides a basis for further clarification of Skemp's (1979) conceptual framework for understanding. Within this conceptual framework, a student may be able to produce a proof, but not understand that it provides generality to the conjecture or that it is a mathematical way of communicating the validity of the conjecture to others. Thus, it makes sense to investigate students' formal understanding of proof in isolation of instrumental and relational understanding. However, further development of a framework for understanding mathematical proof might include possible relationships among the three constructs. While researchers (Hiebert & Lefevre, 1986) have investigated possible relationships between instrumental and relational understanding, formal understanding has been absent from this body of literature.

Methods

The design chosen for this study was an integration of a naturalistic paradigm into an investigation of the students' cognitive views of the purposes of geometric proof, which acknowledges the potential progress of the students' understanding rather than their deficiencies (Lincoln & Guba, 1985; Moschkovich & Brenner, 2000). The study utilized multiple data sources (described in Gfeller, 2004) to inform the research question: field notes and transcripts of classroom observations from participant observation, a preconceptions questionnaire, a post-instruction questionnaire, journal questions, informal and formal interviews with the students, examinations of curriculum materials, homework assignments, and assessment tools, and a researcher journal.

The study was conducted over a period of approximately four months. The students' views were contextualized through classroom observations, conducted in the style of participant-as-observer (Spradley, 1980). The classroom observations were used to a) develop instruments for data collection such as journal questions and a post-instruction questionnaire, b) capture the subtleties and nuances of the meanings expressed by the classroom teacher and students, and c) collect data on the students' views through direct observations as they worked in small groups. The classroom consisted of 19 students and their teacher; 15 agreed to participate with a majority of the participants at the sophomore level. The classroom teacher, Mrs. Kelly, had taught high school mathematics for eight years at this public school.

A combination of analyses was conducted: analytic induction, constant comparison, and typology. On-going analyses of the field notes obtained from the classroom observations and documents were used to inform the interviews. As new data were collected, changes in categories and perceptions were made (Miles & Huberman, 1994).

Students' Views of the Purposes of Proof

In general, the students in the study experienced difficulty in expressing their views of the purposes of geometric proof. At the completion of the unit on geometric proofs, students were asked to list as many purposes of geometric proof. Only four out of fifteen students were able to give at least one purpose of proof. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked during an interview to describe the purpose for each column, all of the students referred to both explanation and verification.

The students also expressed limited views of the purposes of proof, referring mainly to verification. Only a few students referred to explanation, systematization, and communication when describing geometric proofs on journal questions and the post-instruction questionnaire. However, students typically referred to at least two of purposes of proof (explanation, verification, and communication) when describing the proving process involved in coordinate geometry.

Finally, the students' views of various purposes of geometric proof were diverse. Descriptions of the students' views of geometric proof are provided according to categories established by de Villiers: verification, explanation, communication, systematization, and discovery.

Verification

Verification is the process for establishing the validity, or truth, of a geometric statement. The verification process for geometric proof involves conviction in the validity of a statement established through experimentation or intuition as well as the presentation of a proof. The judgment of the lengths of line segments prior to geometric proof is one manifestation of intuition (Burton, 1999; de Villiers, 1999; Hersh, 1993). In terms of verification as a process for establishing the validity of a statement, approximately one-half the students in the study reported verification as the main purpose of two-column geometric proof. Classroom observations also indicated that establishing the validity of a statement in geometry through proofs was a consistent theme supported by Mrs. Kelly throughout the two units. During instruction on coordinate geometry proofs, Mrs. Kelly established a direct relationship between the algebraic portion of a proof and verification. Similarly, the *Statements* column of a geometric proof was the designated "area" for showing all valid statements in a proof. In addition, responses from the journal questions indicated that the students' views mirrored the same relationships that were established through the instruction regarding this aspect.

The investigation of the students' views on the relationship between conviction and verification was difficult because it was understood by most of the students that all statements they were attempting to prove were true. The classroom activities lacked "cognitive unity of theorem" meaning that no experimentation or conjecturing was required by the student (Garuti, Boreo, Lemut, & Mariotti, 1996). Thus, the investigation of the students' views regarding the relationship between gaining conviction and establishing validity was examined by asking students about using intuition in the proving process. Fischbein (1982) would have called this form of intuition "intuitive acceptability."

Four students (Lori, Seth, Mackenzie, and Nikki) were sure that intuition should not be used in proving statements in geometry. To make sure that students considered the whole process of proving, these students were asked to consider the used of intuition from beginning to end. However, all four students stood by their original view that intuition should not be used in the proving process:

Researcher: Do you think proving a statement in geometry involves intuition? Why or why not? And you said no because the lines may look like they are the same, you have to prove it. Nothing is proven from intuition. Well, what is your definition of intuition?

Seth: That you can feel that it is right.

Researcher: Do you ever have that feeling when you are doing a proof?

Seth: No.

Researcher: So when you look at a proof and it says 'given' and 'prove', did you think that it might not be true, the prove statement?

Seth: No.

Researcher: Did you think they were always true?

Seth: No, but that's not intuition. They're either true or they're not true. You don't get feelings about whether you think it’s right or not. Because you can either look at it and it looks right but it could still be wrong. See? Or, it could look wrong and actually be right. Because lines mean absolutely nothing, it's just whether or not it's been proven.

Researcher: Okay. Do you think anyone uses intuition?

Seth: I'm sure people do, but (pause)

Researcher: But they shouldn't?

Seth: They're probably not always right.

Two students, Ken and Penny, expressed uncertainty about using intuition in proof writing, but made sure to explain that intuition did not prove anything. Nine of the remaining students believed that intuition is used in writing geometry proofs. However, their descriptions of how intuition is used in the proving process indicated two different processes: judgment and “preformal” proving. Blum and Kirsh (1991) defined preformal proving as “a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises” (p. 187). Five of these students (Betty, Brandi, Cathy, Kara, and Marty) specifically mentioned that judgment of a diagram played a role in proving geometry. For example:

> You have to look at the picture and have a feeling about something that would help you to solve the problem. Then you can get to where you need to be. (Kim, Item 4, Post-instruction Questionnaire)

Cathy also seemed to include preformal proving as part of her description of intuition. Four other students (Betty, Brandi, Kara, and Marty) also seemed to view intuition as part of a preformal proving process. For example:

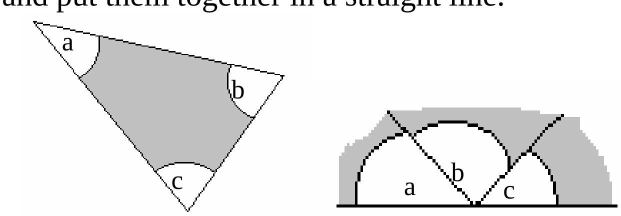
> You need to have some idea of what you have to do in order to complete the task of solving the proof. (Betty, Item 4, Post-instruction Questionnaire)

Explanation

Proof as explanation refers to revealing the underlying mechanism why a statement in geometry is true. The properties of geometric figures were the underlying mechanism for the coordinate geometry proofs while the congruent triangle theorems were the underlying mechanisms for two-column geometric proofs in this study. The students' first exposure to the explanatory feature of geometric proof by the classroom teacher was a colloquial view, telling someone "why you did what you did if you got into trouble." Since the colloquial definition of explanation failed to mention that the reasons for an explanation must pertain to a "corpus of reference" (Douek, 2000), students were not given the opportunity to view explanation in mathematical proof separately from explanation in everyday argumentation.

Students' views of explanation were mainly examined through an item on the pre-instruction questionnaire called the Cardboard Triangle Activity (see Figure 1). Prior to instruction, students who commented on the argument typically equated the word "proof" with verification rather than explanation when asked to assess the given argument.

Suppose your math teacher held up a huge triangle made from cardboard. Then, she tore off the angles and put them together in a straight line.



Then she said that there are 180^{o} in a straight line and when you add the interior angles of any triangle your answer is always 180^{o}.

Do you think your teacher proved that the sum of the interior angles of any triangle is 180^{o}? Why or why not?

Figure 1. Cardboard Triangle Activity

During individual interviews, students were asked to examine their prior response to the argument. When asked whether the fictitious teacher's proof explained why there were 180 degrees in a triangle, four students (Eric, Jeremy, Marty, and Betty) maintained that it was a good explanation. However, the remaining three students (Cathy, Ken, and Lori) believed that the argument presented by the fictitious teacher did not explain why the angles in a triangle added up to 180 degrees. Ken was not sure why he thought the argument was not a good explanation while Cathy and Lori thought the fictitious teacher should have offered more mathematical explanations:

Researcher: Now, everything you know about this word [proved] now, after having done two column proofs and you are going to re-evaluate this situation, would you stay with what you wrote or would you switch it?

Cathy: I would switch it because it's not really proven. When you look at that, it may look like 180 degrees but it might not be exactly. Like, it's not proven.

Researcher: So why doesn't it meet the qualifications for a proof?

Cathy: Because, um, (pause). It doesn't have any statements or reasons?

Researcher: It doesn't have any statements or reason? Okay, well, I could argue that it does have reasons because the reason was that she put them in a straight line. Is that a reason?

Cathy: Yeah, (pause). Basically she's just drawing a picture and playing with it. There, it's a 180 degrees but it (pause) I don't how to (pause)

Researcher: You don't know how to word it?

Cathy: Yeah! (expressing a sigh of relief)

Researcher: When you originally did this, what did you think that word meant? In what sense were you using the word proof? (pause) Can you think back and think why you said yes?

> Cathy: Because, um, she drew a picture and, or not drew a picture, but did that, and um, she was like showing how it would be, but she didn't really do it, (pause) mathematically.

Since Cathy seemed frustrated by not being able to put into words what she was thinking, she was not asked to explain further what she meant by “mathematically.”

Communication

Proof as a means of communication was introduced on the first day of the coordinate geometry unit. Mrs. Kelly’s introductory statement about the topic focused on communication:

> We are going to raise the bar, raise the *standard* on your communication skills and your thought processes, and we are going to ask you to think a little bit deeper, and use the knowledge you already have. (Classroom Observation, 3/6/03)

Students typically stated that coordinate geometry proofs are written so that others may understand it. This view was, in general, not expressed by the students when discussing the purpose of two-column proofs. Two factors may have contributed to the differences in views regarding communication between coordinate geometry proofs and two-column proofs. First, proof as a means of communication to others was consistently addressed by Mrs. Kelly during the coordinate geometry unit, yet mentioned only once at the end of the unit on two-column proofs. Second, coordinate geometry proofs were written in paragraph form while two-column proofs were written as steps.

In both units, Mrs. Kelly established standards of communication. However, the standards of communication for coordinate geometry proofs and two-column proofs were different. For example, when referring to coordinate geometry proofs Mrs. Kelly stated:

> Pretend that you are explaining to, not really to someone in your class, but explaining to someone a couple of years younger than you. Not someone in kindergarten, but someone who has a basic understanding of the math that you are doing, but maybe not all of it, maybe not as much as you know. (Classroom Observation, 3/6/03)

For two column proofs, Mrs. Kelly insisted on the following: a) proofs are to be done in two columns, b) skipping steps was not allowed, and c) reasons should be written in the conditional form if possible. Even though standards of communication were given, it was never stated to the students how these standards are accepted by the mathematical community (National Council of Teachers of Mathematics, 2000).

When students were asked to explain how proof is a form of communication, three categories emerged: a) Demonstration of Ideas, b) Exchange of Ideas, and c) Other. For first category, *Demonstration of Ideas*, eight students (Cathy, Jeremy, Ken, Kim, Marty, Penny, Seth, and Sue) seemed to view proof as a demonstration of knowledge about geometry. For example:

> You’re showing what you're thinking and why you're thinking it so when people look at them they know what was going through your mind when you were trying to solve it. (Kim, Item 5, Post-instruction Questionnaire)

For the second category, *Exchange of Ideas*, four students (Betty, Brandi, Kara, and Nikki) seemed to view proof as an exchange of ideas between people. For example:

> You have to talk and inquire if it’s all of what you think or part of it. So you have to get what other people are thinking to get the full statement. (Brandi, Item 5, Post-instruction Questionnaire)

The group work that occurred during several of the classes throughout the two units seemed to instill in these students that proof writing was an activity that consisted of “helping” rather than “arguing.” This view ran somewhat contradictory to the communication that was observed from the group work produced by the Ken, Marty, and

Jeremy, but consistent with the observations of Betty, Lori, and Sue. When working in small groups, Ken, Marty, and Jeremy argued frequently about the conclusions that could be drawn as well as the meanings of the symbols used in the diagrams. For the third category, *Other*, Lori and Mackenzie simply stated that proofs show (and explain) how statements are true. Eric did not respond.

<u>Systematization</u>

Unfortunately, systematization was first introduced to the students about six months prior to the inception of this study. The meaning of mathematical definition, axioms, postulates and theorems were only briefly reviewed during the unit on two-column geometric proofs through an introductory statement about proofs. Throughout the unit of two-column proofs, Mrs. Kelly mentioned only a few times that theorems had to be proven before they could be used as reasons in another proof, although some theorems were used without proof. In naïve ways, two students (Cathy and Kim) mentioned systematization, but in a less global sense than described by de Villiers (1999):

> They (proofs) tell you what to do and how to do it. They tell you what order to do problems in. They make you think about some things that your mind just normally skips over. (Kim, Item 2, Post-instruction Questionnaire)

During the individual interview, Kim explained why one should not skip over steps.

Researcher: Why can't you just skip over things?

Kim: It makes you think about how your mind works. You might need some information later in the proof.

Kim's response was interesting because, even though Mrs. Kelly mentioned in class that steps should not be skipped, she did not announce to the class any particular reason. However, during an informal interview, Mrs. Kelly mentioned that she had warned some students about making "half-circular" arguments on homework problems. Mrs. Kelly described "half-circular" arguments as proofs that contained reasons that have not yet been derived from the situation.

Discovery

The students' views of proof as a means of discovery generally extended only to the discovery of new methods in mathematics and new mathematical statements. Due to the structure of the homework exercises on geometric proofs, none of the students expressed the view that geometric proof is useful in discovering the underlying mechanism for a geometric situation. Discovery was not discussed by the classroom teacher during either of the two units on proofs. On the post-instruction questionnaire, students were asked to describe how proof is used for discovery. Nine of the students described discovery in terms of new knowledge and were generally thinking about the "prove" statement (conjecture). For example:

> Different theorems are results of someone's life work to prove something. Everything you do in math could form a new result. (Lori, Item 6, Post-instruction Questionnaire)

Four of the students also mentioned the discovery of new methods of proof. For example:

> You could find a new way in discovering steps for geometry proofs. (Marty, Item 6, Post-instruction Questionnaire)

Two other students, Seth and Brandi, did not provide meaningful answers. Seth simply stated that statements are true when they are proven, and Brandi said that she did not know how to answer the question. Since students mainly used the triangle congruency method in constructing proofs, it was not unusual to find that none of the students mentioned discovery in the context of discovering the underlying mechanism of a geometry situation. Three students (Cathy, Seth, and Nikki) were asked during their interview whether they believed they had discovered anything by constructing geometry proofs. All three students said that they did not feel like they discovered anything during the two units on proof. Seth's reason was that one could not discover what the teacher already knew. He seemed to be thinking in terms of public rather than private discovery. Nikki felt that because the same underlying principle had been applied to all of the problems that were assigned, no discovery occurred. Nikki also thought that specific information ascertained from constructing proofs was not useful.

Implications and Recommendations for Future Research

The descriptions of the students' views, along with the context of their learning, lead to several implications relating to teaching and learning. These implications provide recommendations regarding the focus for future research in the area of students' formal understanding. The following implications and recommendations for teaching and learning are made.

First, several students experienced difficulty in expressing their views of the purposes of geometric proof when asked directly. One-third of the students could only list properties or theorems they encountered during the unit on geometric proof. However, when these students were asked to describe the purpose for each column, all of the students listed both explanation and verification. In addition, students who referred to verification when asked directly added explanation to their responses when asked to describe the purpose for each column. When describing the process of developing a coordinate geometry proof, twice as many students mentioned verification and explanation, even though coordinate geometry proofs were also taught as two components. Thus, the students' representation of the mathematical object (Dubinsky, 1991; Sfard, 1991) of geometric proof has been formed in a limited manner. It is possible that the students' representation of coordinate geometry proofs, that were written in narrative form, were different from their representation of Euclidean geometry proofs. Thus, future research should explore the possible differences in students' views about the purposes of geometric proof when using paragraph form and when using two columns.

Second, the students' views of two-column geometric proof consisted mainly of verification, which indicated a limited view of the purpose of geometric proof. Prior to instruction of the two units, students related proof as a means of verification. During instruction, the students often interchanged proof and verification, which seemed to indicate that proof as verification had been deeply-rooted in the students' conception of proof. Since most students are exposed to the word "prove" at an early age in everyday situations prior to learning formal mathematical proof in the mathematics classroom, more research is needed to learn about how students can learn the distinctions between everyday proof and mathematical proof. Instruction to broaden the students' views of the purposes of geometric proof probably requires more than simply stating in the classroom other purposes of proof. While Mrs. Kelly stated that the purpose of the second column was to show the reasons a statement was true, proof as explanation was not continued throughout the unit. Instead, Mrs. Kelly focused on the production of a sequence of logically connected statements, which was the main objective for the state assessment. Examinations of students' views of the purposes of geometric proof in classrooms using dynamic geometry software in conjunction with "cognitive unity of theorem" should be conducted to investigate whether this form of instruction will broaden students' views.

Third, the findings indicated that students typically did not view the reasons for a proof as members of an axiomatic system. The students' first exposure to the explanatory feature of geometric proof was a colloquial view, telling someone "why you did what you did if you got into trouble." Since the colloquial definition of explanation failed to mention that the reasons for an explanation must pertain to a "corpus of reference" (Douek, 2000), students were not given the opportunity to view explanation in mathematical proof separately from explanation in everyday argumentation. The students' acceptance of the argument shown in the Cardboard Triangle Activity problem after the two units on proof indicated that the students were followed the sociomathematical norms (Yackel & Cobb, 1996) that had been established by the classroom teacher regarding explanation. Thus, future investigations should examine how teachers can use problems like the Cardboard Triangle Activity as a way for students to explore the difference between explanation in everyday argumentation and mathematical explanation.

Fourth, future studies should explore various relationships among various purposes of geometric proof. Even though students did not typically express relationships between purposes of geometric proof, one student expressed two distinct relationships. Cathy related explanation and systematization in her response about the Cardboard Triangle Activity problem. She also indicated a relationship between explanation and verification by stating that she was more convinced that a statement was true after she saw the explanation. Another relationship between two purposes of proof that exist but was not expressed by the students was that of discovery and systematization. Future investigations should explore whether more mature mathematics students or mathematics teachers view the purposes as related concepts.

Fifth, the surprising finding that almost half of the students in the study had indicated that intuition was involved in the proving process was encouraging. Students described intuition as visual judgment and preformal proving. Both areas should be further investigated to explore the concepts in more detail. More research is needed to explore instruction regarding visual judgment in geometry. In the area of preformal proving, to what extent do students use preformal proving in geometry? How can instruction enhance the use of preformal proving? Do students use preformal proving in proofs involving algebraic concepts?

Finally, in light of these results, more attention to the purposes of proof should be included in current reform documents. For example, *Principles and Standards for School Mathematics* (NCTM, 2000) explicated the importance of explanation, verification, and communication as purposes as geometric proof at the high school level. Explanation and verification are included in the discussion on reasoning while it has been recommended that proof be considered an “accepted method of communication” in the classroom. However, some improvements could help teachers and teacher educators in refining the purposes of proof in the high school classroom and teacher preparation programs. First, the expansion of the purposes of proof to include systematization and discovery is recommended. Systematization should be explained not only in terms of definitions, axioms, postulates, and theorems, but also in terms of its usefulness in recognizing inconsistencies, circular arguments, and missing links in reasoning. Discussion about proof as discovery should focus on how conclusions of proofs can be used in other proofs, which was clearly not realized by the students in this study. Second, teaching episodes should be developed and included that show connections between various purposes of proofs. For example, the Cardboard Triangle Activity could be used to show students specific criteria for explanation, in a mathematical sense. In addition, the difference between argumentation, justification, and explanation in mathematics should be more clearly defined. Teachers should pay special attention to the differences among the various terms expressed in *Principles and Standards for School Mathematics* (NCTM, 2000) as well as everyday meanings and mathematical meanings in the classroom.

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