Chapter 1

Finite Geometries

**Goal:** In this chapter we will discuss the axiomatic method and use it to develop various finite geometries. We will also define incidence, projective and affine geometries.

1.1 The Axiomatic Method

The *Axiomatic Method* is a procedure by which we prove that results discovered by experimentation, observation, trial and error or even intuition are indeed correct.

The *proof* of a result is a sequence of statements, each of which follows logically from the ones before and leads from a statement that is known to be true to the statement which is to be proven.

**Definition 1.1.1** The axiomatic method consists of

1. A set of technical terms that are chosen as undefined and are subject to the interpretation of the reader.

2. All other technical terms are defined by means of the undefined terms. These are the definitions of the system.

3. A set of statements dealing with undefined terms and definitions that are chosen to remain unproven. These are the axioms of the system.

4. All other statements of the system must be logical consequences of the axioms. These are the theorems of the system.

By giving each undefined term in the system a particular meaning, we create an interpretation of the system. If for a given interpretation of a system, all of the axioms are “correct” statements we call the interpretation a *model*.

1. *Concrete* models have interpretations of the undefined terms adapted from the real world.
2. **Abstract** models have interpretations of the undefined terms taken from some other axiomatic system such as the real number system.

### 1.1.1 Properties of Axiomatic Systems

1. **Consistency:** A set of axioms is consistent if it is impossible to deduce from the axioms a theorem that contradicts any axiom or previously proven theorem. Equivalently, an axiomatic system is inconsistent if it implies a contradiction, that is, if it is possible to prove in it that some statement is both true and false.

**Example:**
The following axiomatic system is inconsistent: *(Any one of the axioms can be proven false using the other axioms.)*

- \( A_1 \). There are exactly 2 boys.
- \( A_2 \). There are exactly 3 girls.
- \( A_3 \). Each boy likes exactly 2 girls.
- \( A_4 \). No two boys like the same girl.

To establish the *absolute consistency* of a set of axioms you need to produce a concrete model.

**Example:** The following axiomatic system is consistent:

- \( A_1 \). There are at least two houses.
- \( A_2 \). For any two distinct houses, there is exactly one street connecting them.
- \( A_3 \). No street connects all houses.
- \( A_4 \). Given any street \( S \) and any house \( H \) not on \( S \), there is exactly one street on which \( H \) lies, but none of the houses on \( S \) lie on this street.

Producing a concrete model to establish absolute consistency is not always possible. Alternatively, we can establish the *relative consistency* by producing an abstract model.

2. **Independence:** An axiom is independent if it cannot be logically deduced from the other axioms in the system. The entire set of axioms is independent if each axiom is independent.

To show that an axiom is independent you produce a model in which that axiom is incorrect and the rest of the axioms are correct. Remember from logic that only correct statements may be logically deduced from correct statements.
3. **Completeness**: An axiomatic system is complete if it is impossible to add an additional consistent and independent axiom without adding additional undefined terms.

It is often quite difficult to prove that a set of axioms is complete. An easier approach is to prove that the set of axioms is categorical which means that each of its models is isomorphic to every other model. It is a proven result, that if a system of axioms is categorical, then it is complete. Recall that, two models are isomorphic if there is a 1-1 correspondence between the elements of the models which preserves all relations existing in either model.

**Exercise**: Show that the following system is not categorical. (*Hint: Show that the system has two completely different models*)

- **A1**: There exist five points
- **A2**: Each line is a subset of those five points.
- **A3**: There exist two lines
- **A4**: Each line contain at least two points.

**Exercise**: Show that the following system is categorical. (*Hint: Show that the system has exactly one model*)

- **A1**: There exist exactly three distinct points.
- **A2**: Any two distinct points have exactly one line on them.
- **A3**: Not all points are on the same line.
- **A4**: Any two distinct lines have at least one point in common.

**Exercise**: Consider a set of undefined elements, \( S \), and the undefined relation \( R \) which satisfies the following axioms:

- **A1**: If \( a \in S \) then \( a R a \).
- **A2**: If \( a, b \in S \) and \( a R b \) then \( b Ra \).
- **A3**: If \( a, b, c \in S \), \( a R b \), and \( b R c \), then \( a R c \).
  1. Devise a concrete model for the system.
  2. Devise an abstract model for the system.
  3. Are the axioms independent?

**Solution**:

1. Let \( S \) be the set of students on the Cortland campus. Let \( R \) be the relation “has the same hair color” or “has the same weight”.
2. Let $S$ be the set of triangles. Let $R$ be the relation “is congruent to” or “is similar to”.

3. To show that the axioms are independent, consider the following:
   • $A_1$ is independent: Let $S = \{x, y, z\}$ and $R = \{(x, x), (y, y), (x, y), (y, x)\}$. Clearly the relation $R$ on $S$ satisfies axioms $A_2$ and $A_3$ but not $A_1$. A second example could be: Let $S = \{Jim, Amy, Mary\}$ and $R = \{(Jim, Jim), (Amy, Amy), (Amy, Jim), (Jim, Amy)\}$ where $(x, y)$ means: “$x$ likes $y$”.
   • $A_2$ is independent: Let $S$ be the set of real numbers, and let $R$ be the relation “is less than or equal to”.
   • $A_3$ is independent: Let $S$ be the set of students on the Cortland campus and let $R$ be the relation “is an acquaintance of”.

1.2 Finite Geometries

**Definition 1.2.1** In order for a configuration of points and lines to be considered a finite geometry, several properties should be satisfied. These include finiteness, uniformity, uniqueness, and existence. The desirable properties are:

1. The number of points is finite.
2. The number of lines is finite.
3. Each line is on the same number $s$ of points, where $s \geq 2$.
4. Each point is on the same number $t$ of lines, where $t \geq 2$.
5. Each pair of distinct points is on at most one line.
6. Each pair of distinct lines is on at most one point.
7. Not all points are on the same line.
8. There exists at least one line.

**The Four-Point Geometry:**
The undefined terms are *point*, *line* and *on*. The axioms are:

• $A_1$. There exists exactly four points.
• $A_2$. Any two distinct points have exactly one line on both of them.
• $A_3$. Each line is on exactly two points.

**Theorem 1.2.1** The Four-Point geometry has exactly six lines.
Proof. By \( A_1 \) we have \( \binom{4}{2} = \frac{4 \cdot 3}{2!} = 6 \) pairs of distinct points. By \( A_2 \) the geometry has at least 6 lines. Assume there is a 7th line. By \( A_3 \) this line has to have two points on it. If one of these points is not one of the original 4 points then \( A_1 \) is contradicted, otherwise \( A_2 \) is contradicted.

Theorem 1.2.2 In the Four-Point geometry, each line has exactly one line parallel to it.

Proof. By \( A_2 \) each point in the geometry is connected to each of the other points by a line. Hence each point has at least three lines on it. Assume a point has 4 lines on it. This 4th line cannot be on any of the other three points without contradicting \( A_2 \). By \( A_3 \) this line has two points on it so the geometry has at least 5 points in it which contradicts \( A_1 \). Hence each point has exactly 3 lines on it. Now given a line \( l \), it has two points on it, say \( P_1 \) and \( P_2 \). By \( A_1 \) there is a point \( P_3 \) not on \( l \). Now this point has three lines on it, and by \( A_2 \) two of them intersect \( l \). Hence \( l \) has at least one line parallel to it. Assume there is a second line parallel to \( l \). This line is not on \( P_3 \) since it already has three lines on it. It also is not on \( P_1 \) or \( P_2 \), hence either this line has one point on it which contradicts \( A_2 \) or has two points on it which contradicts \( A_1 \). Hence each line in the geometry has exactly one line parallel to it.

**Fano’s Geometry:**
The undefined terms are *point*, *line* and *on*. The axioms are:

- \( A_1 \). There exists at least one line.
- \( A_2 \). There are exactly three points on every line.
- \( A_3 \). Not all points are on the same line.
- \( A_4 \). There is exactly one line on any two distinct points.
- \( A_5 \). There is at least one point on any two distinct lines.
Definition 1.2.2 The plane-dual of a geometry is obtained by interchanging the words “line” and “point” in the axioms determining the geometry. A slight adjustment in the grammar might be needed.

Homework 1.2.1 1. Define the plane dual of the Four-Point geometry. This is called the Four-Line geometry. Make a drawing representing the Four-Line geometry.

2. The Five-Point geometry: Consider the following Axioms:
   - A1. There exists exactly five points.
   - A2. Any two distinct points have exactly one line on both of them.
   - A3. Each line is on exactly two points.
     (a) Make a drawing representing the Five-Point geometry.
     (b) State and prove two theorems in the Five-Point geometry.
     (c) Make a drawing representing the plane dual of the Five-Point geometry.

3. Write the plane dual of the axioms of Fano’s geometry and draw a representation of a model satisfying these axioms.

1.3 Incidence Geometry

The undefined terms are point, line and on. The axioms are:
   - A1. For each two distinct points there exists a unique line on both of them.
   - A2. For every line there exist at least two distinct points on it.
   - A3. There exists at least three distinct points.
   - A4. Not all points lie on the same line.
A geometry that satisfies all four axioms will be called an *incidence geometry*. An incidence geometry has one of the following parallel postulates:

**Parallel Postulates:** Given a line \( l \) and a point \( P \) not on \( l \), then three possibilities exist for a parallel axiom:

- **\( PP_1 \):** There exist no lines on \( P \) that are parallel to \( l \).
- **\( PP_2 \):** There exists exactly one line on \( P \) that is parallel to \( l \), or
- **\( PP_3 \):** There exists more than one line on \( P \) parallel to \( l \).

An incidence geometry satisfying \( PP_2 \) is said to be Euclidean, otherwise it is said to be non-Euclidean.

**Definition 1.3.1** A projective geometry is an incidence geometry having no parallel lines (parallel postulate 1) and in which each line has at least three points.

The Four-Line, Five-line geometries and Fano’s geometry are projective.

**Definition 1.3.2** An affine geometry is an incidence geometry that exhibits the Euclidean parallel postulate.

**Homework 1.3.1** 1. Determine which of the following interpretations of the undefined terms are models of incidence geometry and indicate which parallel postulate is exhibited.

(a) Points are points on a Euclidean plane, and lines are nondegenerate circles in the Euclidean plane.

(b) Points are points on a Euclidean plane, and lines are all those lines on the plane that pass through a given fixed point \( P \).

(c) Points are points on a Euclidean plane, and lines are concentric circles all having the same fixed center.

(d) Points are points in the interior of a fixed circle, and lines are parts of straight lines that intersect the interior of the circle.

(e) In Euclidean 3-space, points are interpreted as lines and lines as planes.

(f) Points are lines in Euclidean 3-space, and lines are planes all of which contain the same fixed line.

**Kirkman’s Schoolgirl Problem:**

This was posed by the Rev. Kirkman in 1850: A school mistress has fifteen girl pupils and she wishes to take them on a daily walk. The girls are to walk in five
rows of three girls each. It is required that no two girls should walk in the same row more than once per week. Can this be done?

For more detail see Projective Space Walk For Kirkman’s Schoolgirls by Sr. Rita Ehrmann, Mathematics Teacher, volume 68, 1975, pages 64 - 69.

Bibliography