

# TRANSFORMATION GEOMETRY AND THE ARTWORK OF M. C. ESCHER

*Students can learn to analyze the symmetries and transformations in Escher's drawings. What's more, they can be taught the techniques necessary to make their own drawings.*

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IN RECENT years the name of M. C. Escher, the Dutch artist (1898-1972), has been gaining fame in many mathematics classrooms. This is certainly justified and long overdue, since Escher's works are those of a skilled mathematician.

With the rising popularity of transformation geometry, a study of Escher's "regular division of the plane" is tailor-made for the geometry classroom. Note Escher's own reference to transformations (Escher 1973, p. 9):

Anyone who wishes to achieve symmetry on a flat surface must take account of three fundamental principles of crystallography: repeated shifting (translation); turning about axes (rotation) and gliding mirror image (reflexion). [parentheses his]

Escher's artwork is extremely popular with students. A lesson on symmetry can culminate nicely with a study of the symmetry of his artwork. The symmetries studied and found in Escher's "regular division of the plane" are reflection symmetry, rotation symmetry, translation symmetry, and glide-reflection symmetry. Only a very basic understanding of a symmetry transformation is needed for the investigation of the artwork. Most students have an intuitive grasp of the concepts, making the presentation of definitions, such as this one, optional:

*A figure  $\alpha$  is symmetric if and only if there exists some transformation  $S$ , not the identity transformation, such that  $S(\alpha) = \alpha$ .*

Figures 8 through 13 are used with permission of the Escher Foundation, Haags Gemeentemuseum, The Hague.

## Symmetric Figures

Each of figures 1 through 7 exhibits at least one of the four types of symmetries. Examples such as these quickly convey the basic principles of symmetry to students. Many find it easy to produce their own examples.

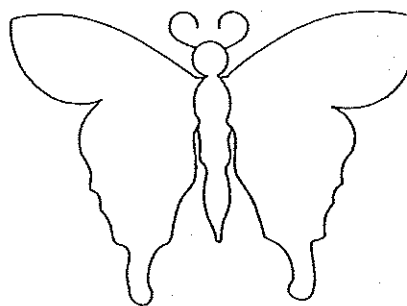


Fig. 1. Reflection symmetry

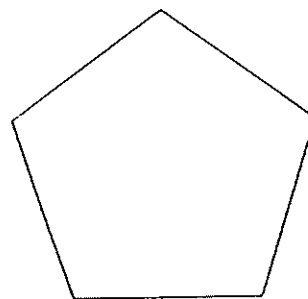


Fig. 2. Reflection symmetry and rotation symmetry

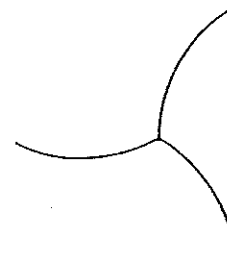


Fig. 3. Rotation symmetry

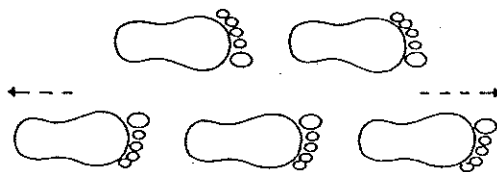


Fig. 4. Translation symmetry and glide-reflection symmetry

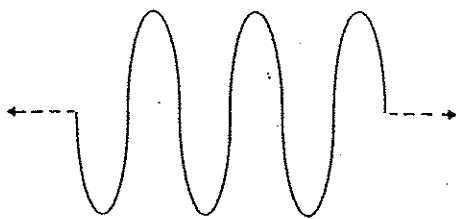


Fig. 5. All four types of symmetries

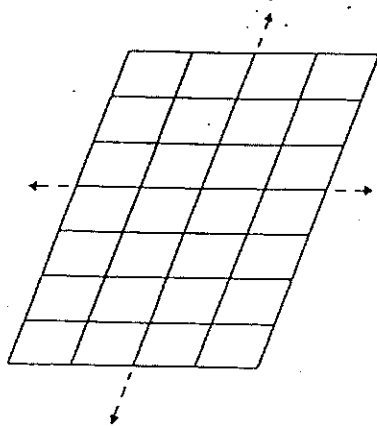


Fig. 6. Rotation symmetry and translation symmetry

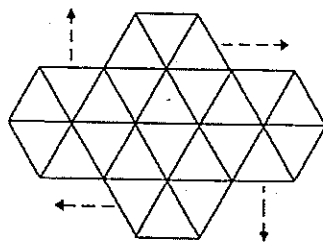


Fig. 7. All four types of symmetries. Notice that there are rotational symmetries of  $60^\circ$ ,  $120^\circ$ , and  $180^\circ$ . Several nonparallel symmetry lines and infinite nonparallel directions of translation can also be identified.

For figures 4 through 7 it must be imagined that the figures continue without end in the directions indicated by the dashed arrows. This is necessary whenever one is dealing with translation and glide-reflection symmetries. When certain polygons, such as a parallelogram or an equilateral triangle, tessellate a plane, a variety of symmetries are exhibited (figs. 6, 7).

### Escher's Symmetry Drawings

The figures in Escher's "regular division of the plane" are derived from polygons that tessellate a plane. The pictures contain some, but not all, of the symmetries exhibited by the original (polygonal) tessellation. Most students easily recognize the symmetry in Escher's pictures.

The congruent flying fish illustrated in figure 8 are derived from equilateral triangles. There is translation symmetry and rotation symmetry, with centers  $A$ ,  $B$ , and  $C$ .  $B$  is not a center if the gray fish may not rotate onto a white one.



Fig. 8. M. C. Escher's *Symmetry Drawing No. 99*

Squares are the basis of the tadpoles in figure 9. Investigate the centers of rotation. Translation symmetry is also exhibited.

A different arrangement of triangles yields the swans in figure 10. Glide reflection symmetry is displayed.

The lizards in figure 11 began as regular hexagons and exhibit both rotation and translation symmetries.

Figures 12 and 13 are two examples of

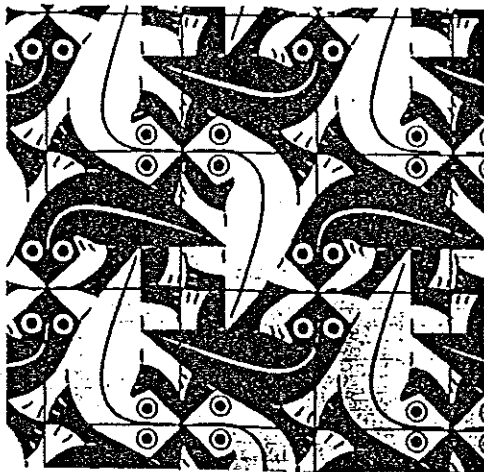


Fig. 9. M. C. Escher's *Symmetry Drawing No. 104*



Fig. 11. M. C. Escher's *Drawing*. Pencil, Black Ink, and Water-colour

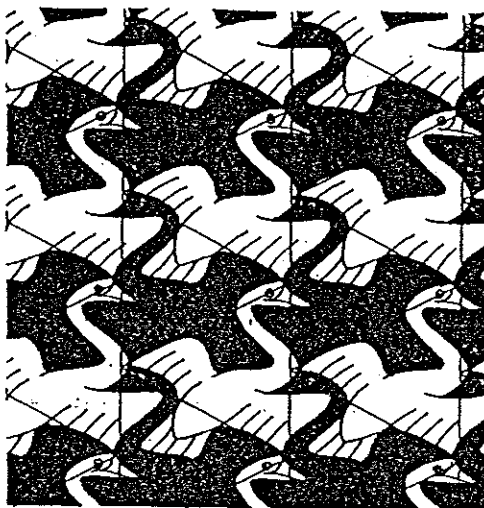


Fig. 10. M. C. Escher's *Study of Regular Division of the Plane with Birds, 1955* (black ink and water-colour in grey-brown)

Escher's other works. Students may enjoy investigating these pictures for the symmetry they exhibit.

With just a little exposure, it becomes fairly easy to identify the symmetries in Escher's artwork. There are a good number of examples that can be studied with the aid of transparencies or slides made from the two collections of Escher's artwork listed at the end of this article (Escher 1971; 1973). Both these publications are paperback, suitable for dismantling. The prints make fine Thermofax transparencies.

#### Designing an Original Symmetry Pattern

After an investigation of the symmetries of Escher's artwork, specifically, his "regular division of the plane," the next logical step is to explore the methods of producing original pictures of that type. Joseph L. Teeters has a fine article in the April 1974 issue of the *Mathematics Teacher* on producing pictures possessing glide-reflection symmetry.

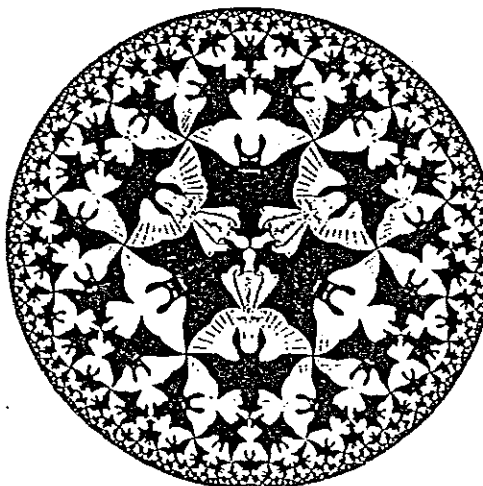


Fig. 12. M. C. Escher's *Circle Limit IV 1960* (woodcut)

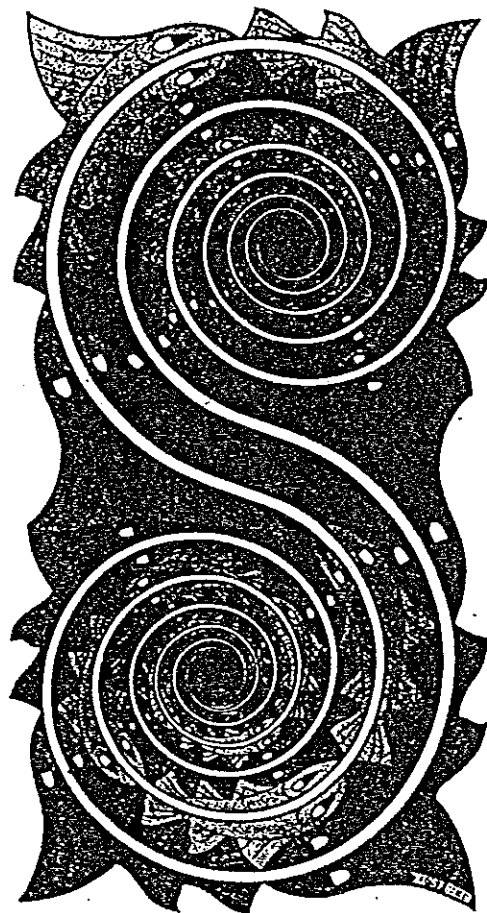


Fig. 13. M. C. Escher's *Whirlpools 1957* (woodcut)

The present article focuses on a method used to produce drawings exhibiting any of the four types of symmetries. The specific directions produce a pattern with translation and rotation symmetries.

Escher based his patterns on congruent polygons that tessellate the plane. Any such polygon can be used, but it is generally easier to begin with the square. Here is an approach you can use to generate your own artwork.

- Materials:** one 9 cm × 9 cm square of construction paper  
 scissors  
 tape  
 ruler or compass

Students should label their squares as shown in figure 14. The students should in-

vestigate various transformations that will move the square all over the plane until it is totally covered with squares. As one example

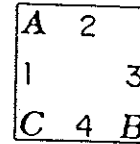


Fig. 14

(fig. 15), square 1 can be rotated four times about point *B*, 90° each time, before it returns to its original position. It can then be translated two squares over, both horizontally and vertically, and the rotations can be repeated. Keep track of where the numbered sides of square 1 land as it moves over the plane. The numbers and letters in figure 15 are arranged vertically for easy reading and are not meant to represent the actual orientation that would result from moving the original figure. Notice that rotations of 90° about *A* and 180° about *C* result. Point *C* need not be labeled.

From this diagram, one can see that sides 1 and 2 must fit together (as in a jigsaw puzzle), as must sides 3 and 4. Point out to your students that point *A* is the common endpoint of sides 1 and 2 and that point *B* is the common endpoint of sides 3 and 4. *A*

|   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 2 | C | 1 | A | A | 2 | C | 1 | A |   |
| 1 | 1 | 3 | 4 | 2 | 1 | 3 | 4 | 2 |   |   |
| C | 4 | B | B | 3 | C | 4 | B | B | 3 |   |
|   | 3 | B | B | 4 | C | 3 | B | B | 4 | C |
| 2 | 4 | 3 | 1 | 2 | □ | 4 | 3 | 1 |   |   |
| A | 1 | C | 2 | A | A | 1 | C | 2 | A |   |
| A | 2 | C | 1 | A | A | 2 | C | 1 | A |   |
| 1 | 3 | 4 | 2 | 1 | 3 | 4 | 2 |   |   |   |
| C | 4 | B | B | 3 | C | 4 | B | B | 3 |   |
|   | 3 | B | B | 4 | C | 3 | B | B | 4 | C |
| 2 | 4 | 3 | 1 | 2 | 4 | 3 | 1 |   |   |   |
| A | 1 | C | 2 | A | A | 1 | C | 2 | A |   |

Fig. 15

"protrusion" on side 1 is a congruent "indentation" on side 2, both equidistant from point *A*. (See square II in fig. 15.) Sides 3 and 4 and point *B* are likewise related.

Have your students cut any shape out of any side of the square, without cutting off a corner (fig. 16). (Later, investigate ways to

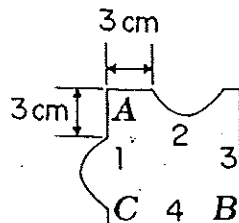


Fig. 16. Corresponding sides: 1 - 2, 3 - 4

cut the corners.) Measure how far away this piece was from the labeled point on that side. (Use point *A* for sides 1 and 2, point *B* for sides 3 and 4.) Now, *without turning the piece over*, reposition it on the corresponding side. It should be taped in position the same distance from point *A* (or *B*) as it was before it was moved.

Continue this process a few more times. Cut into any unaltered part of the square, except a corner. Reposition the cut piece on the corresponding side an equal distance from the common endpoint. (If a numeral is cut off, simply remember which side is which.) The result of the cuts (see fig. 17e) can be shaded in some imaginative way (see fig. 17f). The enlarged figure 17g summarizes the cuts made in figures 17a-17e. Students should be encouraged to exchange their cutouts, and to let their imaginations run wild in deciding what the cutout resembles.

Finally, to produce the entire "Escher design," have each student trace around his or her own cutout at the blackboard or on a large sheet of paper (fig. 18). Rotate the figure 90° about point *B* and trace again. Repeat twice, then translate two squares over, and begin rotating again. Add the final detailing.

With a bit of practice, the students' cutouts look more and more like animals, flowers, and so on.

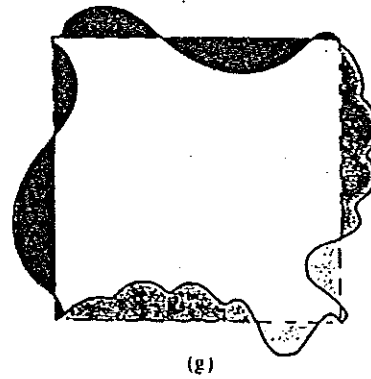
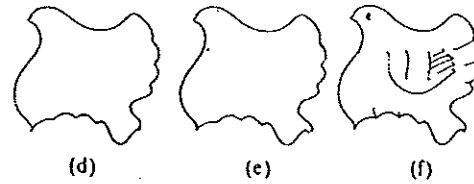
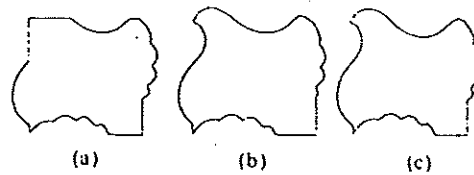


Fig. 17

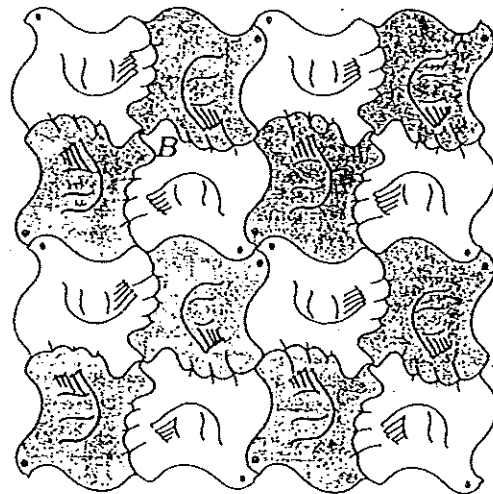


Fig. 18

### Symmetry Patterns Based on Other Polygons

To find the corresponding sides for other polygons that tessellate the plane, in-

investigate the transformations that will move the polygon over the entire plane. Record the new positions of the sides. When cutting and repositioning, measure the piece from the common endpoint of the corresponding sides. If a side corresponds to itself, as is the case with the equilateral triangles, measure from the midpoint of the side. Figures 19, 20, and 21 offer some starting possibilities. As in figure 15, the numbers and letters are arranged vertically for easy reading and are not meant to represent the actual orientation that would result from moving the original figure. Figure 22 is derived from the hexagonal scheme.

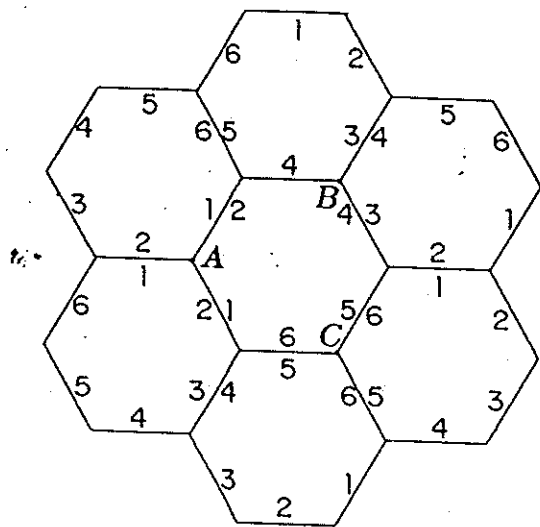


Fig. 19. 1 - 2, 3 - 4, 5 - 6

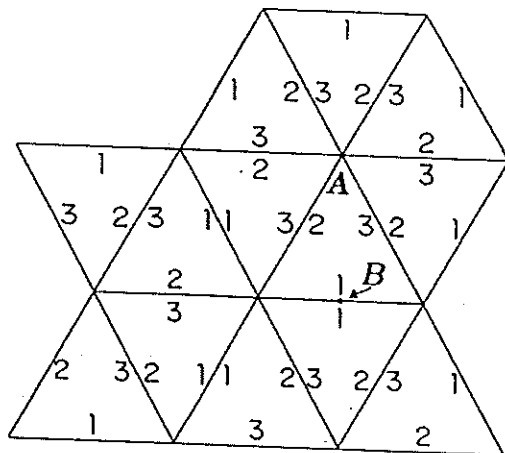


Fig. 20. 1 - 1, 2 - 3

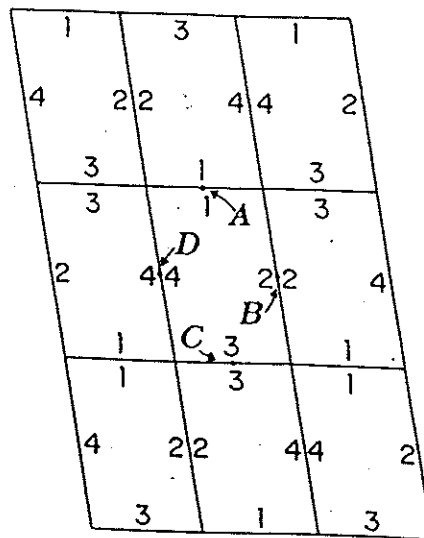


Fig. 21. 1 - 1, 2 - 2, 3 - 3, 4 - 4

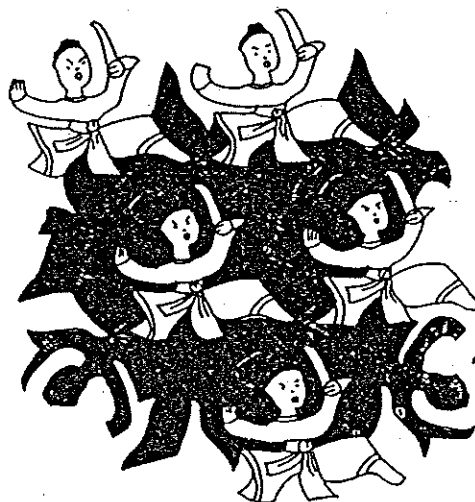


Fig. 22

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