Iterated Function Systems and Fractals

by

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Abstract: The study of fractals arising as attractors of IFS's became an area of practical importance thanks to Mandelbrot's fundamental insight that many natural objects have some self similarity and Barnsley's insight that it is possible to begin with a shape and determine an IFS whose attractor converges onto that shape . A number of artificial as well as natural fractals and their possible IFS's will be investigated using java applets that are freely available on the Web.

Affine transformations:

An affine transformation is of the form

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

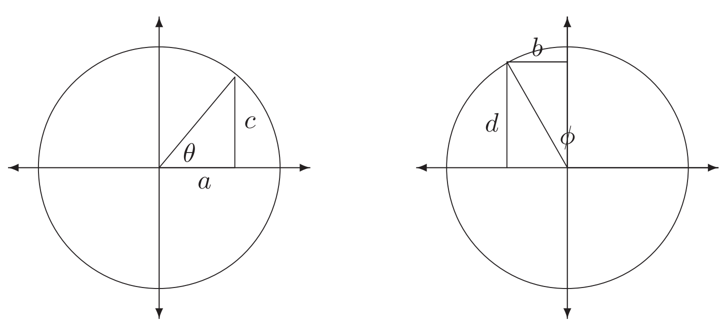
or

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} r\cos(\theta) & -s\sin(\phi) \\ r\sin(\theta) & s\cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

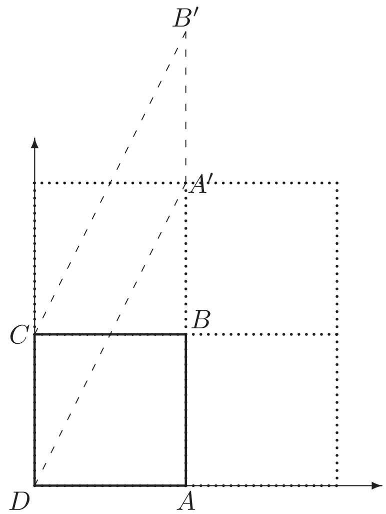
- r and s are the scaling factors in the x and y directions resp.
- θ and ϕ measure rotation of horizontal and vertical lines resp.
- e and f measure horizontal and vertical translations resp.

It can be easily shown that:

- $r^2 = a^2 + c^2$,
- $s^2 = b^2 + d^2$,
- $\theta = \arctan(c/a)$, and
- $\phi = \arctan(-b/d)$.

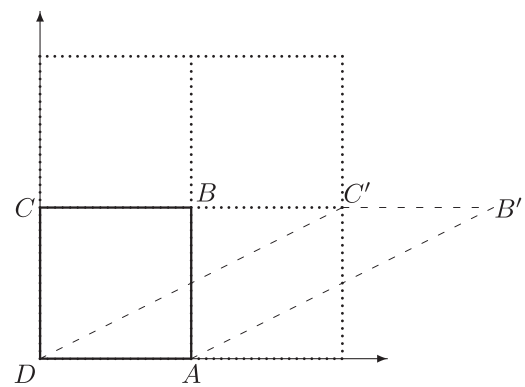


For example, shears are represented as follows:



$x' = x$
$y' = 2x + y$

Shear in the y direction



$x' = x + 2y$
$y' = y$

Shear in the x direction

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$r = \sqrt{5}, \qquad s = 1$$
$$\theta = 63.435, \phi = 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$r = 1, \qquad s = \sqrt{5}$$
$$\theta = 0, \phi = -63.435$$

Finding IFS Rules from Images of Points

Given three non-collinear initial points $p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3)$ and three image points $q_1 = (u_1, v_1), q_2 = (u_2, v_2), q_3 = (u_3, v_3)$ respectively. Find an affine transformation T such that $T(p_1) = q_1, T(p_2) = q_2$, and $T(p_3) = q_3$.

Recall that,

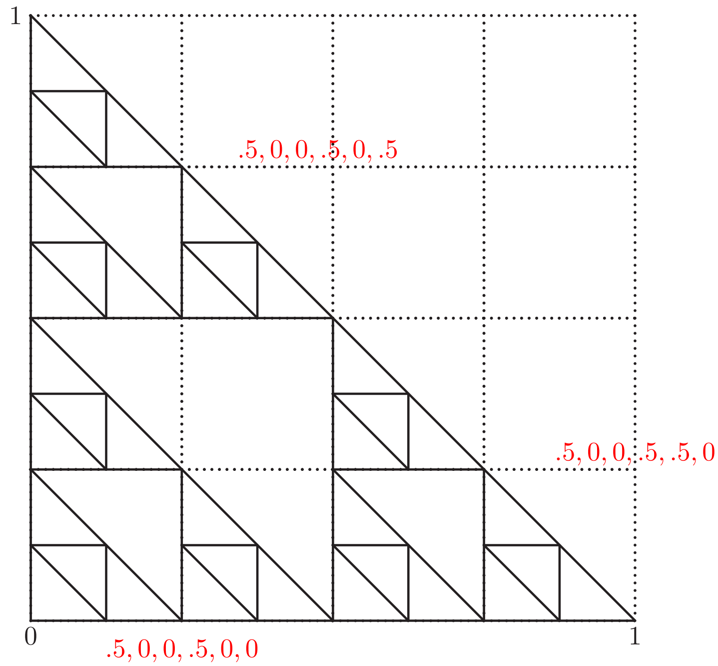
$$T(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Now Using p_1, p_2, p_3 and their images we arrive at the following six equations in six unknowns:

$$\begin{aligned} ax_1 + by_1 + e &= u_1 \\ cx_1 + dy_1 + f &= v_1 \\ ax_2 + by_2 + e &= u_2 \\ cx_2 + dy_2 + f &= v_2 \\ ax_3 + by_3 + e &= u_3 \\ cx_3 + dy_3 + f &= v_3 \end{aligned}$$

This system has a unique solution if and only if the points p_1, p_2, and p_3 are noncollinear.

Example: The Seirpinski Triangle



The IFS for the Seirpinski triangle is $\{T_1, T_2, T_2\}$ where

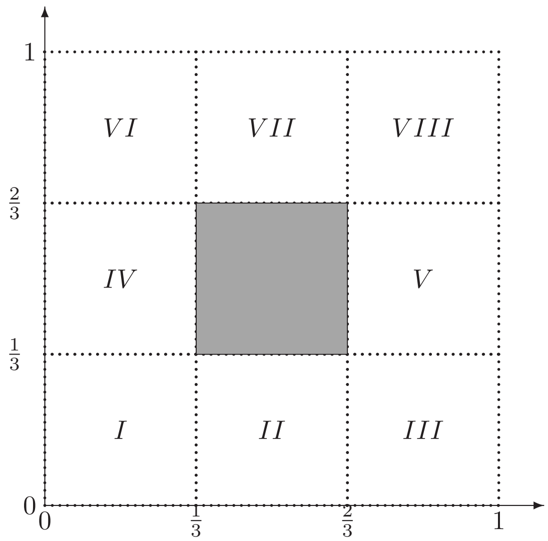
$$T_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .5 \\ 0 \end{pmatrix}$$

$$T_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ .5 \end{pmatrix}$$

http://smccd.net/accounts/hasson/fract.html

Example: The Seirpinski Carpet



The IFS for the Seirpinski carpet is $\{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ where

$$T_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$T_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} ..333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .333 \\ 0 \end{pmatrix}$$

$$T_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .666 \\ 0 \end{pmatrix}$$

$$T_4(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ .333 \end{pmatrix}$$

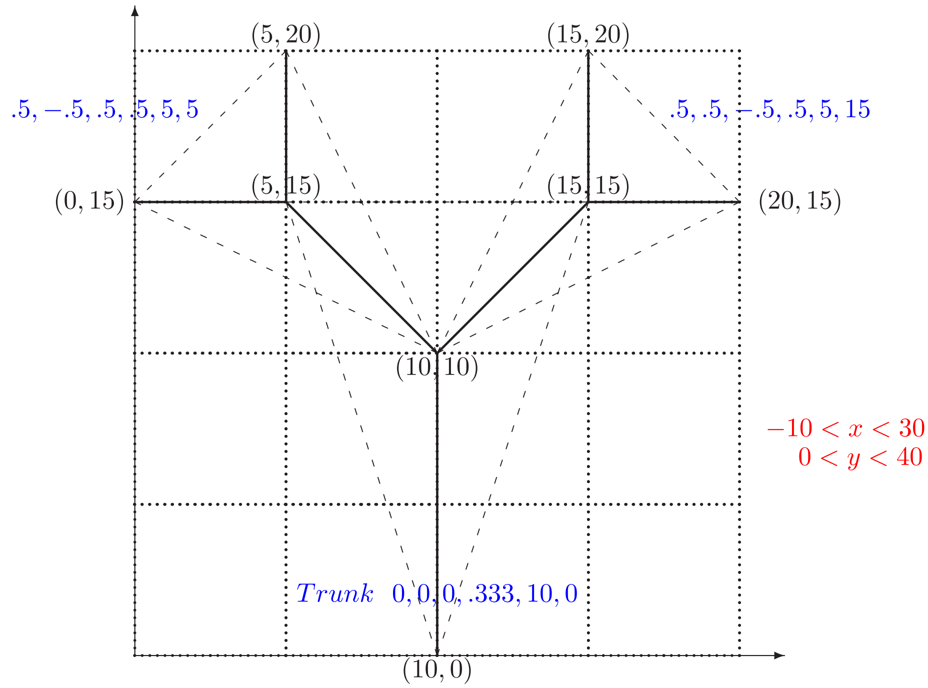
$$T_5(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .666 \\ .333 \end{pmatrix}$$

$$T_6(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ .666 \end{pmatrix}$$

$$T_7(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .333 \\ .666 \end{pmatrix}$$

$$T_8(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .333 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .666 \\ .666 \end{pmatrix}$$

Example: A simple tree



The IFS for the tree is $\{T_1, T_2, T_2\}$ where

$$T_1(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .5 & .5 \\ -.5 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 15 \end{pmatrix}$$

$$T_2(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} .5 & -.5 \\ .5 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

$$T_3(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & .333 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

http://alumnus.caltech.edu/ chamness/equation/equation.html

http://smccd.net/accounts/hasson/fract.html

Demonstrate some of the java applets at

http://classes.yale.edu/fractals/Software/Software.html

Discuss the IFS for the Fern.
Discuss the IFS for the mountain range.
Discuss the IFS for the Wave.

Iterated Function Systems:

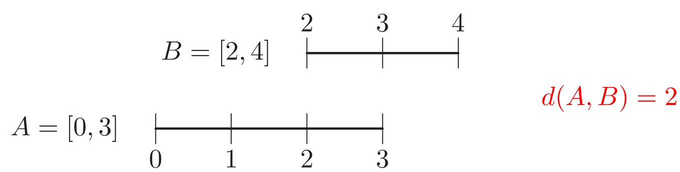
An *Iterated Function System* in R^2 is a collection $\{F_1, F_2, \ldots, F_N\}$ of contraction mappings on R^2. (F_i is a contraction mapping if given x, y then $d(F_i(x), F_i(y)) \leq s\, d(x, y)$ *where* $0 \leq s < 1$)

The Collage Theorem says that there is a unique nonempty compact subset $A \subset R^2$ called the attractor such that

$$A = F_1(A) \cup F_2(A) \cup \ldots \cup F_N(A)$$

Sketch of Proof:

Given a complete metric space X, let $H(X)$ be the collection of nonempty compact subsets of X. Given A and B in $H(X)$ define $d(A, B)$ to be the smallest number r such that each point of in A is within r of some point in B and vice versa. If X is complete so is $H(X)$.



Given an IFS on X, define $G : H(X) \to H(X)$ by

$$G(K) = F_1(K) \cup \ldots \cup F_N(K)$$

If each F_i is a contraction then so is G. The contraction mapping theorem says that a contraction mapping on a complete space has a unique fixed point. Hence there is $A \in H(X)$ such that

$$G(A) = A = F_1(A) \cup F_2(A) \cup \ldots \cup F_N(A)$$

Given $K_0 \subset R^2$, let $K_n = G^n(K_0)$. Then

$$d(A, K_n) \leq \frac{s^n d(K_0, G(K_0))}{1 - s}$$

Hence K_n is a very good approximation of A for large enough n.

To understand the previous result note that:

$$
\begin{aligned}
d(A, K_0) &= d(\lim_{n\to\infty} K_n, K_0) \\
&= \lim_{n\to\infty} d(K_n, K_0) \\
&\leq \lim_{n\to\infty} d(K_0, K_1) + d(K_1, K_2) + \cdots + d(K_{n-1}, K_n) \\
&\leq \lim_{n\to\infty} d(K_0, K_1) + sd(K_0, K_1) + \cdots + s^{n-1} d(K_0, K_1) \\
&\leq \frac{1}{1-s}\, d(K_0, K_1)
\end{aligned}
$$

Hence,

$$d(A, K_1) \leq \frac{1}{1-s}\, d(K_1, K_2) \leq \frac{s}{1-s}\, d(K_0, K_1)$$

$$d(A, K_2) \leq \frac{1}{1-s}\, d(K_2, K_3) \leq \frac{s^2}{1-s}\, d(K_0, K_1)$$

$$\vdots$$

Deterministic Algorithm: (Slow!)

Given K_0, let $K_n = G^n(K_0)$. Now $K_n \to A$ and $n \to \infty$.

Random Algorithm: (Fast!)

Given $Q_0 \in K_0$, let $Q_{i+1} = F_j(Q_i)$ where F_j is chosen with probability p_j from all of the F_i. Clearly $Q_i \in K_i$ and for sufficiently large i, Q_i is arbitrarily close to some point in A.

Choose $p_i \sim \dfrac{|detA_i|}{\sum_{i=1}^{N} |detA_i|} = \dfrac{|a_i d_i - b_i c_i|}{\sum_{i=1}^{N} |a_i d_i - b_i c_i|}$.

Geometrically, choose p_i "proportional to F_i."

How do these algorithms work: