Modeling a Process or a System as a Markov Chain

Jorge Luis Romeu, Ph.D.
Emeritus, State University of New York
Past Director of Region II (NY & PA)
ASQ Reliability Division
Email: Romeu@cortland.edu
http://web.cortland.edu/romeu
June 13, 2024

Summary

Modeling a process or a system as a Markov Chain offers an excellent tool for its performance evaluation and for the study of different factors that can run it astray. In this talk we will present four examples of modeling Covid-19 problems as Markov Chains: (1) the trajectory of Covid-19 infected patients into an ICU, and up to their death; (2) assuming that the virus will infect a large part of the population, thus preventing further community spread and yielding Herd Immunization; (3) study of the Re-opening of Colleges under Covid-19 using a Markov Chain defined over a nine element state space that moves through a set of Transient states, eventually leading to two Absorbing States: Expulsion or Coursework; (4) assessing different patterns of vaccination, which may affect achieving (or not) Herd Immunity: polls suggesting that a significant number of people are not inclined to become vaccinated.

Advantages

Stochastic Modeling allows us to:

- Establish the directions of the transitions
- Obtain the system performance for specified transition rates
- Find appropriate state transition rates to obtain specified system performance
- Play the “what if” game with the system

A Markov Chain model to study the spread of the Covid-19 virus

https://www.researchgate.net/publication/343021113_A_Markov_Chain_Model_for_Covid-19_Survival_Analysis

*This Markov Chain is intended to illustrate the power that Markov modeling offers to Covid-19 studies. This article models the trajectory of Covid-19 infected patients into an ICU, and up totheir death through a Markov Chain. We first consider a simple three-state, recurrent model. Itssteady state probabilities are obtained for an efficient and an inefficient system and we compare.We then include additional absorbing states, to account for more complex situations. Using TPM
we obtain the (1) probability of death of a Patient; and using their sojourns in the different states,(2) their expected time to death. The results are useful in establishing (1) logistic requirements ofhealth care units, to provide excellent patient care, and (2) some objective Triage procedures, ifever such extremes are required. This Markov Chain tutorial has been, by far (i.e. by the numberof hits in its LinkedIn and ResearchGate web pages) our most read Covid-19 stats report.

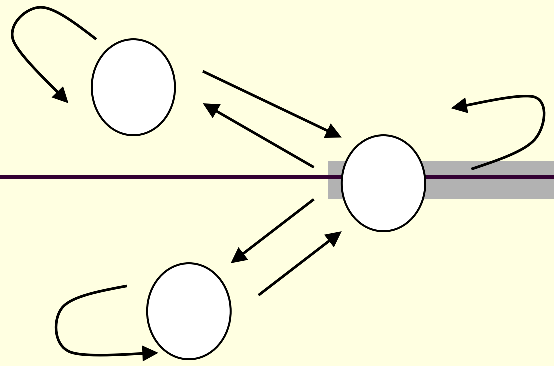
Modifying the Covid-19 Example

- Substitute contexts with industrial examples.
- State Diagrams and rates are the same.
- Calculations and numerical results are same.
- Interpretations now within industrial context.
- Everything else also stands.

First Example: Repairable Systems

- Assume you have a repairable system
- With three recurring states:
  - Up/Full Operation
  - Degraded Operation
  - Down/Failed State
- Assume Device is fixed to Degradable state
  - Starts operating again and continues repair
  - Eventually may become Fully Operational
  - Or may again Fail and cycle repeats
- Time step is One Day

Markov Chain State Space Diagram Over a Simple Three-element state space: (0) Operating, (1) Degraded and (2) Failed.



Transition Probability Matrix P for Three State *Markov Chain*

$$P = \begin{array}{cccc} States & 0 & 1 & 2 \\ 0 & p_{00} & p_{01} & p_{02} \\ 1 & p_{10} & p_{11} & p_{12} \\ 2 & p_{20} & p_{21} & p_{22} \end{array} = \begin{array}{cccc} States & 0 & 1 & 2 \\ 0 & 1-q_{01} & q_{01} & 0 \\ 1 & p_{10} & 1-p_{10}-q_{12} & q_{12} \\ 2 & 0 & p_{21} & 1-p_{21} \end{array}$$

$$\Pi = Limit_{T\to\infty}(\Pr ob\{X(T)=0\}, \Pr ob\{X(T)=1\}, \Pr ob\{X(T)=2\}) = (\Pi_1;\Pi_2;\Pi_3)$$

Comparisons of two-systems performance measures:

Case (Rates)	Long-run	Operating	Degraded	Failed System
Efficient (5%)	Probabilities	0.545	0.273	0.182
Efficient (5%)	Times Between	1.834	3.667	5.50
Inefficient (10%)	Probabilities	0.387	0.322	0.290
Inefficient (10%)	Times Between	2.583	3.099	3.444

Efficient system: *transition probability* (Degraded) is 0.05 and that of remaining Failed (State 2) is 0.7.
Ineficient system: *transition probability* (Degraded) is 0.1 and that of remaining Failed (State 2) is 0.8.

For Efficient System: $T_0 = 1/\pi_0 = 1/0.545 = 1.834$; $T_1 = 1/\pi_1 = 3.667$; $T_2 = 1/\pi_2 = 5.50$;

Second Example: One Shot Device

- Assume you have a working Hospital
- On-Line Electrical Power from Company Service
- If On-Line Electrical Power Fails, then
  - Off-Line Hospital Power Generators take over
  - Two generators provide Full Service
- If One Off-Line Generator fails, Degraded Service
- If Both Generators Fail, final option kicks in:
  - Emergency Generator takes over (Degraded)
- If this generator fails, then System Fails.

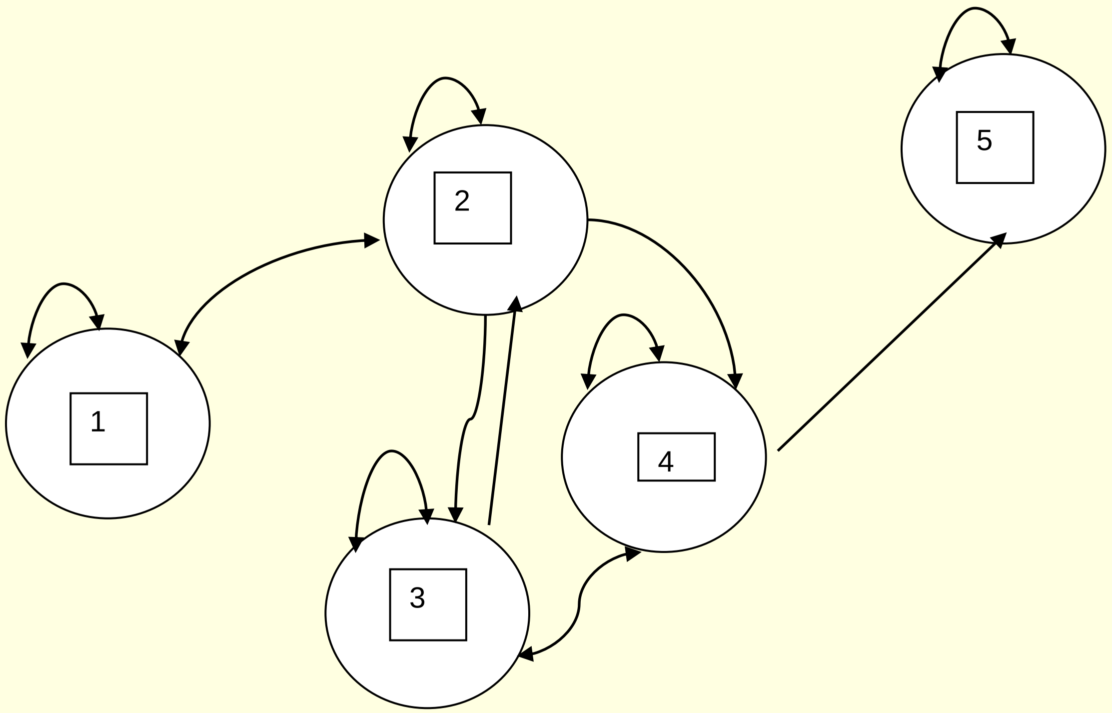
Markov Chain over five states

(1)	(2)	(3)	(4)	(5)	
0.93	0.07	0.00	0.00	0.00	(1); System working On-Line
0.05	0.80	0.10	0.05	0.00	(2): Off-Line, Fully operational (2)
0.00	0.15	0.80	0.05	0.00	(3): Off-Line, Degraded (1 Unit)
0.00	0.00	0.05	0.80	0.15	(4): Emergency Unit only
0.00	0.00	0.00	0.00	1.00	(5): Failed System (absorbing state)

Matrix inverse $(I-Q)^{-1}$ of the Transient States: Sojourns in Trans. States Before Absortion.

(1)	(2)	(3)	(4)
26.1905	16.6667	10.0000	6.66667
11.9048	16.6667	10.0000	6.66667
9.5238	13.3333	13.3333	6.66667
2.3810	3.3333	3.3333	6.66667

Markov Chain State Space Diagram



Average times to Failure from all transient states

Starting State for the System	**Average Time to System Failure**
From Initial System On-Line Time	= 26.19 + 16.66 + 10.00 + 6.66 = 59.52 days
From Off-Line Time (Two Units)	= 16.67 +10 + 6.67 = 33.34 days
From Off-Line Time (One Unit)	= 10.00 + 6.66 = 16.66 days
From the Time of connecting the Emergency Unit only	= 6.66 days
Average Time, System On-Line	= 26.19 days

Probabilities of System Failure, starting from any of the transient states

Starting	State	Two Days	Four Days	Eight Days	Sixteen Days
Initial	On-Line Generator	0.000	0.002	0.018	0.098
From	Off-Line (Two Units)	0.007	0.036	0.118	0.282
From	Off-Line (One Unit)	0.007	0.038	0.127	0.307
From	Emergency Unit only	0.270	0.444	0.636	0.780

Probability of System ever reaching any state from another one

	On-Line	**Off-Line (Two U.)**	**Off-Line (One Unit)**	**Emergency**	**Failed System**
On-Line	0.96	1.0	0.75	1.0	1.0
Off-Line (2 Units)	0.45	0.94	0.75	1.0	1.0
Off-Line (1 Unit)	0.36	0.8	0.92	1.0	1.0
Emergency Unit only	0.09	0.2	0.25	0.85	1.0
Failed	0.0	0.0	0.0	0.0	1.0

A Two-Absorbing-States Markov Chain to study the problem of Covid-19 Herd Immunization

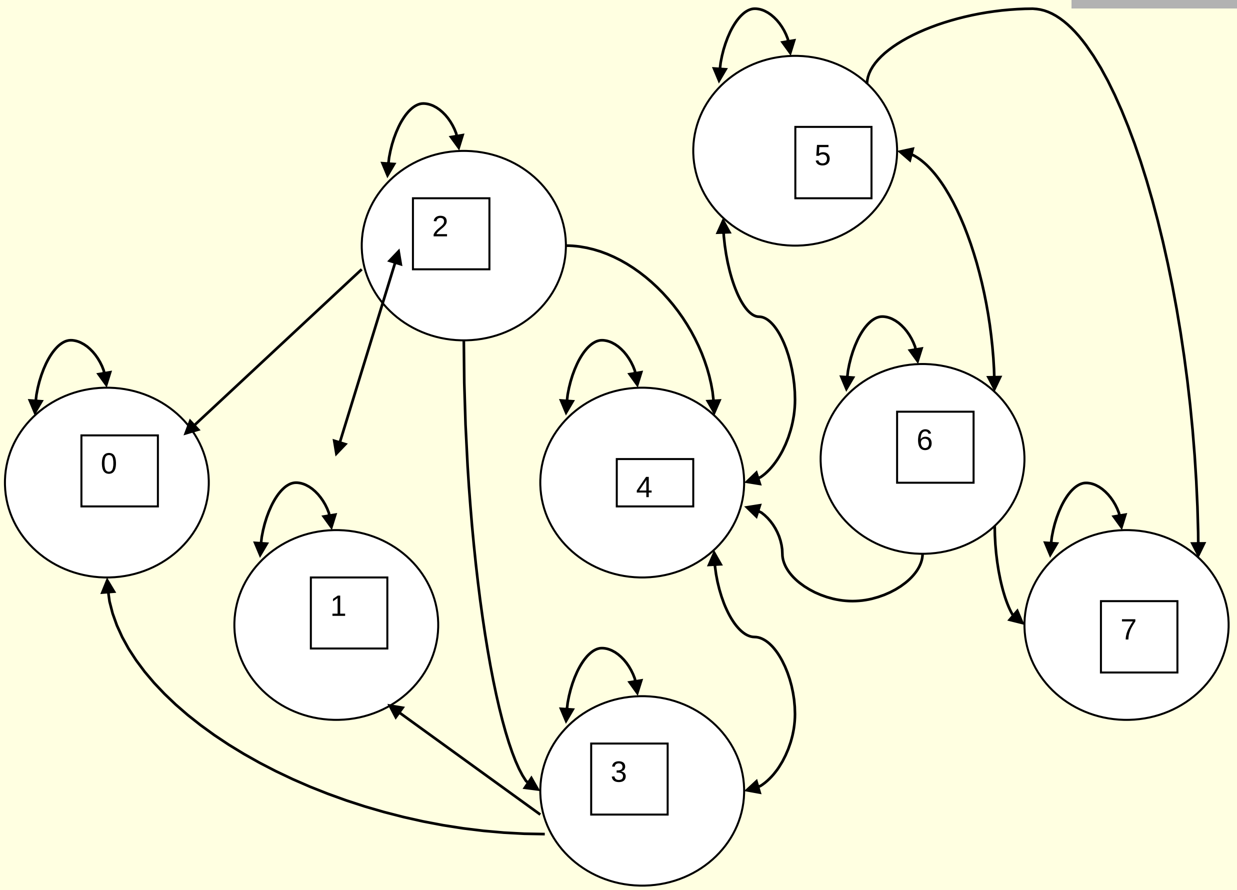
https://www.researchgate.net/publication/343345908_A_Markov_Model_to_Study_Covid-19_Herd_Immunization

This second Markov model assumes that the virus will infect a large part of the population, thus preventing further community spread and yielding Herd Immunization. Our previous Markov Chain assumed there was neither a vaccine nor a treatment for Covid-19. Also that, if current infection rates remained unchecked, everyone would eventually die. This paper assumes thatCovid-19 survivors become immune, thence, cannot become re-infected. There is much debate about employing Herd Immunity as an alternative solution for combating Covid-19. Our Markov Chain quantitatively analyzes such situation. The model obtains (1) the probability of a Patient death or immunization. Also, the (2) expected times to death (or to immunization) when startingfrom different states in the Space (which can be used in Triage situations). Transition rates canhelp compare efficient and inefficient strategies, as well as help establish an acceptable infectionrate. Times spent in a State (Sojourn) help estimate the required size of health care facilities thatwill treat patients. Statistics models help answer many health questions, as well as to compare the performance of different public health strategies, in a more objective, way.

Markov Chain for Herd Immunization
Over an *eight-element State Space*:

(0) Covid-19 Immunized population
(an absorbing state);
(1) *Non Infected* persons in the General Population;
(2) *Infected persons*
(but asymptomatic; i.e. not known to be such);
(3) *Infected persons Detected and Isolated*;
(after symptoms, or Covid-19 tests positive)
(4) *Hospitalized patients*
(after becoming ill with Colvid-19);
(5) *Patients in the ICU* (very sick);
(6) Patients in a Ventilator (critical);
(7) Patient *Death*
(an absorbing state)

Markov Chain *State Space Diagram*



Markov Chain Transition Probability Matrix

State	Immune	UnInfected	Infected	Isolated	Hospital	ICU	Ventilator	Dead
0	1.0	0.00	0.00	0.0	0.0	0.0	0.0	0.0
1	0.0	0.96	0.04	0.0	0.0	0.0	0.0	0.0
2	0.2	0.05	0.40	0.2	0.2	0.0	0.0	0.0
3	0.2	0.05	0.00	0.6	0.2	0.0	0.0	0.0
4	0.0	0.00	0.00	0.1	0.8	0.1	0.0	0.0
5	0.0	0.00	0.00	0.0	0.3	0.3	0.3	0.1
6	0.0	0.00	0.00	0.0	0.1	0.3	0.3	0.3
7	0.0	0.00	0.00	0.0	0.0	0.0	0.0	1.0

Average times to death from each of the transient states

Starting State for any Individual	Average Time to Pass Away (Die)
From Time of Initial Infection (undetected)	=1.667+2.22+5.556+ 0.972+0.417=10.83 days
From the Time of Infection/Isolation	=3.889+5.556+0.972+0.4167 =10.83 days
From the Time of Hospitalization	=11.11+1.94+0.83 = 13.9 days
From the Time of entering an ICU	=2.91667 + 1.25000 = 4.17 days
From the Time of entering a Ventilator	2.08 days

Probability of Dying or Becoming Immune, Starting from a Transient State

Starting State	Probability of Dying	Probability Immunization
Uninfected	0.222222	0.777778
Infected/undetected	0.222222	0.777778
Infected/Isolated	0.222222	0.777778
Hospitalized	0.444444	0.555556
In ICU	0.666667	0.333333
On Ventilator	0.777778	0.222222

A Markov Chain to study the problem of Re-opening Colleges under Covid-19

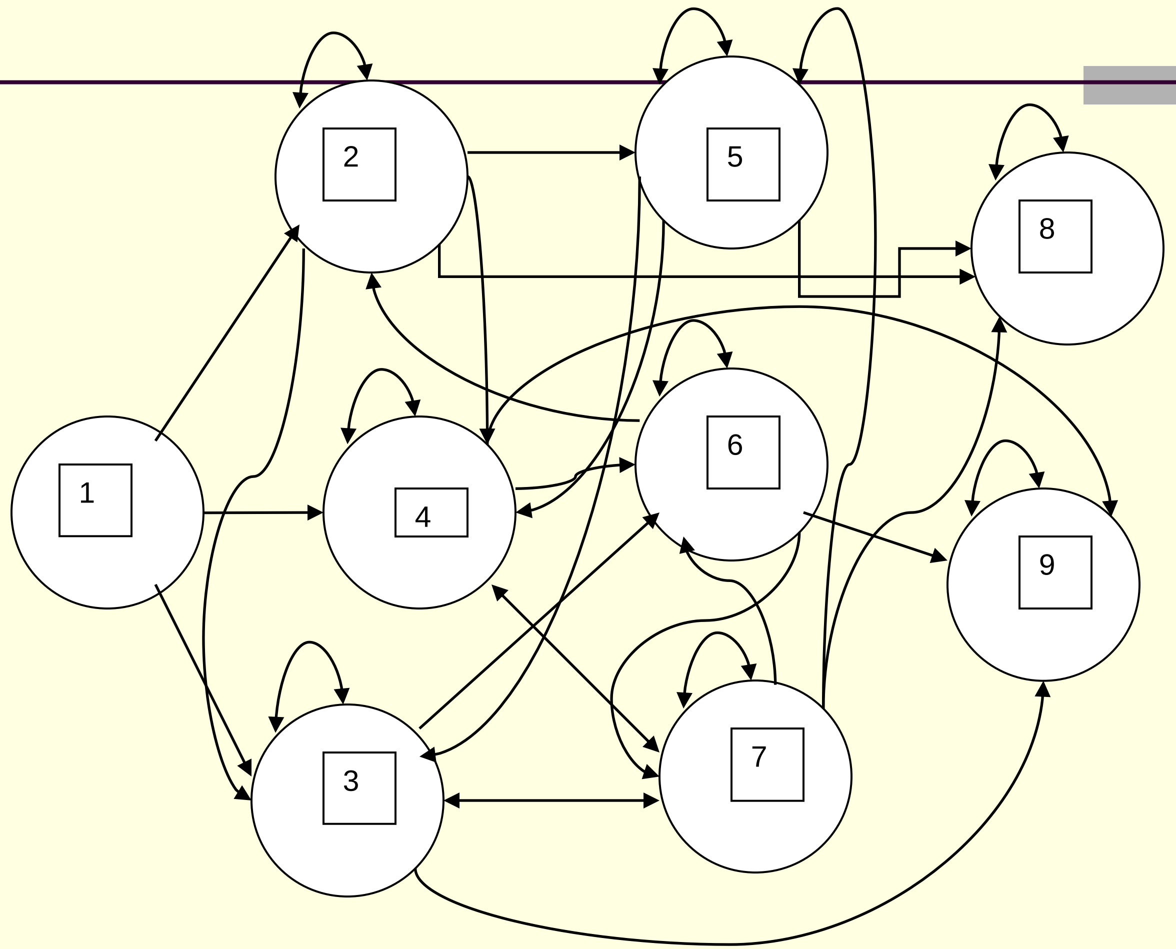
https://www.researchgate.net/publication/343825461_A_Markov_Model_to_Study_College_Re-opening_Under_Covid-19

This Markov Model studies the dilemma of Re-opening Colleges under Covid-19. We analyze the situation using a Markov Chain defined over a nine element state space that moves through aset of Transient states, eventually leading to two Absorbing States: Expulsion or Coursework Completion. The model, due to its specific State Spaces and transition probabilities is very usefulto compare reopening plans. Through the infection (transition) rates we study their impact on the probabilities of Expulsion and Course Completion. Differing infection rates depend on student compliance with community public health measures such as face covering, social distancing, etc.By assigning different values to these rates, their impact can be assessed and compared. Once updated and fine tuned (or rebuilt) Markov models can be used by college authorities to re-assessand improve their reopening plans, by faculty and students, to assess their risks in such openings, and by governments, to assess the validity and safety of such plans, thus allowing or proscribing them.

Markov Chain for College Reopening **over a *nine-element State Space***

(1) *Arrival to Campus* and *Covid- 19 testing*;
(2) *Infected students* go *into Isolation* units;
(3) Some students are placed in *Presential* courses;
(4) Other students are placed in *Distance Learning* courses;
(5) Some *students who violated Code* are *placed in Suspension*;
(6) Some *students* become *infected* with Covid-19,
but are *not detected* as such;
(7) Some *students violate* code but are *not detected*;
(8) *Absorption*: Some students are Expelled from College
(9) *Absorption*: Other students Complete their Semester

Markov Chain *State Space Diagram*



Markov Chain Transition Probability Matrix

St.	1	2	3	4	5	6	7	8	9.
1	0	0.05	0.35	0.6	0.00	0.00	0.00	0.00	0.0
2	0	0.70	0.20	0.0	0.05	0.00	0.00	0.05	0.0
3	0	0.00	0.80	0.0	0.00	0.05	0.05	0.00	0.1
4	0	0.00	0.00	0.8	0.00	0.05	0.05	0.00	0.1
5	0	0.00	0.10	0.1	0.70	0.00	0.00	0.10	0.0
6	0	0.50	0.00	0.0	0.00	0.30	0.00	0.00	0.2
7	0	0.00	0.00	0.0	0.70	0.10	0.00	0.20	0.0
8	0	0.00	0.00	0.0	0.00	0.00	0.00	1.00	0.0
9	0	0.00	0.00	0.0	0.00	0.00	0.00	0.00	1.0

Probability of Expulsion or Completion, Starting from a Transient State

Starting State	Probability of Expulsion	Probability of Completion
Arrival	0.216839	0.783161
Infected	0.384030	0.615970
Presential	0.208039	0.791961
Distance Learning	0.208039	0.791961
Suspension	0.472026	0.527974
Infected but undetected	0.274307	0.725693

A Markov Model to Assess Covid-19 Vaccine Herd Immunization Patterns

https://www.researchgate.net/publication/347441411_A_Markov_Model_to_Assess_Covid-19_Vacine_Herd_Inmunization_Patterns

The Markov model assesses different patterns of vaccination, which may affect achieving (ornot) Herd Immunity. The urgency of this paper stems from polls suggesting a significant number of people are not willing to become vaccinated. Herd Immunity can be acquired by (a) letting thevirus infect most of the population. Weaker ones (the elderly, those with co-morbidities etc.) willdie) and those surviving will become immunized; alternatively, (b) by vaccinating a large part ofthe general population. Vaccination carries two aspects: one individual and the other social. First, the vaccine protects the individual. Secondly, if enough individuals in the general population are vaccinated, the activity has an effect over the Pandemic. With few new customers to infect, the virus starves and disappears. By changing the vaccination parameters (e.g. infection rates, participation and immunization percentages), model results will differ, allowing the comparison of different public health strategies.

Conclusions

Markov Models can be improved by modifying their state space, transition rates and/or transition directions, among other modifications.

Alternatively, models can be used as an illustration or to assess and improve engineering plans, or to assess their risks when participating in such *plans*,

Changing transition rates in the Markov Chain helps study how *they affect absorption probabilities* and thus *compare efficient and inefficient plans*, as well as to *help establish an acceptable community infection rate*