<u>Tutorial: Branch and Bound Algorithm</u>

For the Resolution of Integer Linear Programs:

MGS411 – Intro to Management Sciences
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<u>Modification of Example of Ch. 7, Page 321</u>

Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
End

<u>First</u>: this is the solution of the Relaxed (standard) Linear Program:

Optimal solution found at step: 1
Objective value: 14.65116

Variable	Value	Reduced Cost
X1	**2.441860**	**0.0000000E+00**
X2	**3.255814**	**0.0000000E+00**

<u>Notice that this solution is NOT INTEGER</u>

Row	Slack or Surplus	Dual Price
1	14.65116	1.000000
2	0.0000000E+00	0.1013715E-01
3	0.0000000E+00	0.5813953E-02
4	1.558140	0.0000000E+00

Objective Coefficient Ranges

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	2.000000	0.1428571	1.700000
X2	3.000000	17.00000	0.2000000

Righthand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	1365.000	261.3000	409.5000
3	140.0000	60.00000	38.28571
4	4.000000	INFINITY	1.558140

We now follow the Branch and Bound Algorithm to find the Integer Solution.

First, notice that the Relaxed solution (14.6) is an Upper Bound for the Integer Solution.

<u>Then, take the Variable whose Non Integer value is further away from an Integer:</u>

In the present case this is **Variable X2 = 2.441860**;

BREAK IN VARIABLE X1: TWO OR LESS AND THREE OR MORE

First, obtain the solution for the Branch X1 <= 2

```
Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
x1 <= 2;
End
```

Optimal solution found at step: 0
Objective value: 13.90000

Variable	Value	Reduced Cost
X1	2.000000	0.0000000E+00
X2	3.300000	0.0000000E+00

Row	Slack or Surplus	Dual Price
1	13.90000	1.000000
2	74.10000	0.0000000E+00
3	0.0000000E+00	0.7500000E-01
4	2.000000	0.0000000E+00
5	0.0000000E+00	1.700000

Ranges in which the basis is unchanged:

Objective Coefficient Ranges

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	2.000000	INFINITY	1.700000
X2	3.000000	17.00000	3.000000

Righthand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	1365.000	INFINITY	74.10000
3	140.0000	10.85714	132.0000
4	4.000000	INFINITY	2.000000
5	2.000000	0.4418605	2.000000

Now, notice that the Relaxed solution (13.9) is smaller than the Upper Bound obtained before.

Also, compare the values of the Ranges obtained here, and previously.

Finally, use next **THREE OR MORE AND NOT GREATER THAN TWO.**

<u>Obtain the solution for the Branch X1 > 3</u>

Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
x1 >= 3;
End

Optimal solution found at step: 0
Objective value: 14.57143

Variable	Value	Reduced Cost
X1	**3.000000**	**0.0000000E+00**
X2	**2.857143**	**0.0000000E+00**

Row	Slack or Surplus	Dual Price
1	14.57143	1.000000
2	0.0000000E+00	0.1098901E-01
3	13.71429	0.0000000E+00
4	1.000000	0.0000000E+00
5	0.0000000E+00	-0.1428571

Ranges in which the basis is unchanged:

Objective Coefficient Ranges

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	2.000000	0.1428571	INFINITY
X2	3.000000	INFINITY	0.2000000

Righthand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	1365.000	93.60000	780.0000
3	140.0000	INFINITY	13.71429
4	4.000000	INFINITY	1.000000
5	3.000000	1.000000	0.5581396

Notice how the Relaxed solution (14.57) is smaller than the Upper Bound obtained initially, but larger than the solution obtained in the alternative Branch. (13.9).

Hence, these two solutions constitute the new Lower and Upper Bounds for the problem solution.

The final solution of the Integer Problem should be between these two new values.

If we use **Branch X1 <= 2** we will not obtain any solution greater than 13.9 (Upper Bound there).

Select **Branch X1>=3** to continue your work, and **<u>break in Variable X2 = 2.857143</u>**

BREAK IN VARIABLE X2: TWO OR LESS AND THREE OR MORE

First, obtain the solution for the Branch X2 <= 2

```
Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
x2 <= 2;
End
```

Optimal solution found at step: 0
Objective value: 14.00000

Variable	Value	Reduced Cost
X1	**4.000000**	**0.0000000**
X2	**2.000000**	**0.0000000**

RESULT IS INTEGER, HENCE IT IS A FEASIBLE SOLUTION

Row	Slack or Surplus	Dual Price
1	14.00000	1.000000
2	39.00000	0.0000000E+00
3	44.00000	0.0000000E+00
4	0.0000000E+00	2.000000
5	0.0000000E+00	3.000000

Ranges in which the basis is unchanged:

Objective Coefficient Ranges

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	2.000000	INFINITY	2.000000
X2	3.000000	INFINITY	3.000000

Righthand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	1365.000	INFINITY	39.00000
3	140.0000	INFINITY	44.00000
4	4.000000	0.2000000	4.000000
5	2.000000	0.1428571	2.000000

We have obtained a feasible (Integer) solution through this Branch. We need to analyze the Alternative Branch, in order to decide (1) to continue, or (2) to stop here and take this solution.

<u>Thence, we obtain the solution for the Branch X2 >= 3</u>

```
Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
x2 >= 3;
End
```

Optimal solution found at step: 1
Objective value: 14.65116

Variable	Value	Reduced Cost
X1	**2.441860**	**0.0000000E+00**
X2	**3.255814**	**0.0000000E+00**

NON INTEGER, THENCE INFEASIBLE SOLUTION.

Row	Slack or Surplus	Dual Price
1	14.65116	1.000000
2	0.0000000E+00	0.1013715E-01
3	0.0000000E+00	0.5813953E-02
4	1.558140	0.0000000E+00
5	0.2558140	0.0000000E+00

Ranges in which the basis is unchanged:

Objective Coefficient Ranges

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X1	2.000000	0.1428571	1.700000
X2	3.000000	17.00000	0.2000000

Righthand Side Ranges

Row	Current RHS	Allowable Increase	Allowable Decrease
2	1365.000	261.3000	409.5000
3	140.0000	60.00000	8.800000
4	4.000000	INFINITY	1.558140
5	3.000000	0.2558140	INFINITY

We have arrived to the end of our work, as no other solution will be better (Larger) than the one we have obtained above (14.0).

For completion, we will use LINGO to obtain the Integer Problem Solution, directly:

<u>LINGO INTEGER SOLUTION, DIRECTLY:</u>

```
Model:
Max = 2*x1 + 3*x2;
!Subject to;
195*x1 + 273*x2 <= 1365;
4*x1 + 40*x2 <= 140;
x1 <= 4;
@GIN(x1);
@GIN(x2);
End
```

Notice the way LINGO requires the Integer Constraints to be given.

<u>Output for the Integer Solution:</u>

Optimal solution found at step: 4
Objective value: 14.00000
Branch count: 0

Variable	Value	Reduced Cost
X1	**4.000000**	**-0.5000000**
X2	**2.000000**	**0.0000000**

Row	Slack or Surplus	Dual Price
1	14.00000	1.000000
2	39.00000	0.0000000E+00
3	44.00000	0.0000000E+00
4	0.0000000E+00	0.0000000E+00

Compare with the solution obtained above, using Branch and Bound.

Also, notice that for Integer Problems, directly, LINGO does NOT allow range analyses.

Use this Tutorial to help you develop your Homework.