Tutorial Linear Problem:

A trucking company is currently using two sizes of containers. One type (x) is a large, 4 cubic feet (cu-ft.) container weighing 10 pounds (lb). The second type (y) is a small 2 cu-ft. container weighing 8 lb. Company trucks can handle a maximum load of 3280 lb., and can haul up to 1000 cu-ft. of containers. The company charges 50 cents for each large container and 30 cents for each small one. Determine, using LP, what is the best combination of containers to maximize profit.

Describe the Objective:
Maximize total profit of hauling crates x (large) and y (small)
Max $.50x + .30y$

Describe each Constraint:
Constraint 1: $10x + 8y \leq 3280$ (Maximum weight must be LESS THAN 3280 lb)
Constraint 2: $4x + 2y \leq 1000$ (Maximum space must be LESS THAN 1000 cu-ft)

Nonnegativity Constraints:
$x, y \geq 0$ (Cannot haul a negative amount of containers)

Full Mathematical Model:
Max $.50x + .30y$
Subject to (s.t)

$10x + 8y \leq 3280$	Constraint 1 – maximum load
$4x + 2y \leq 1000$	Constraint 2 – maximum space
$x, y \geq 0$	

Question 1:

Draw the two dimensional LP problem (i.e. the constraints and Objective Function (OF) on the plane. Shade the Feasible Region, determine the coordinates of each corner, and evaluate in each the OF. Select the best corner (optimal solution).

Calculating the Coordinates:

- **Constraint 1:**

  $10x + 8y \leq 3280$

  $10x + 8y = 3280$

  <u>Solve for x = 0</u>

  $10(0) + 8y = 3280$

  $0 + 8y = 3280$

  $y = 3280/8$

  $y = 410$

- **Coordinates for Constraint 1:**

  (0, 410) and (328, 0)

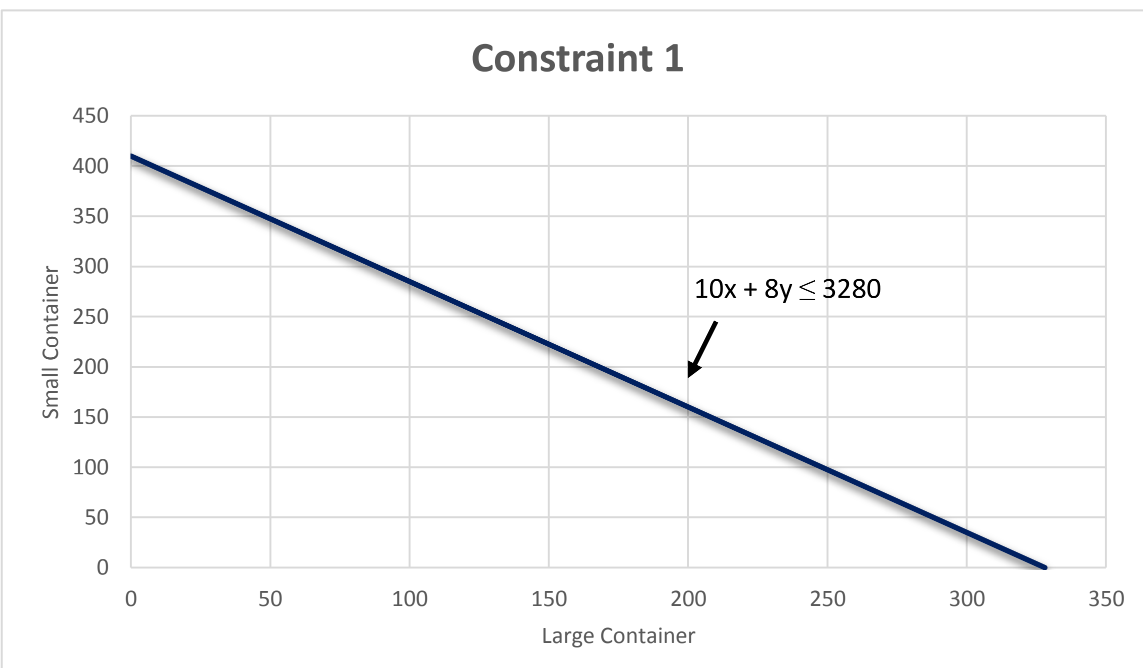
  <u>Solve for y = 0</u>

  $10x + 8(0) = 3280$

  $10x + 0 = 3280$

  $x = 3280/10$

  $x = 328$



- **Constraint 2:**

  $4x + 2y \leq 1000$

  $4x + 2y = 1000$

  <u>Solve for x = 0</u>

  $4(0) + 2y = 1000$

  $0 + 2y = 1000$

  $y = 1000/2$

  $y = 500$

- **Coordinates for Constraint 2:**

  (0, 500) and (250, 0)

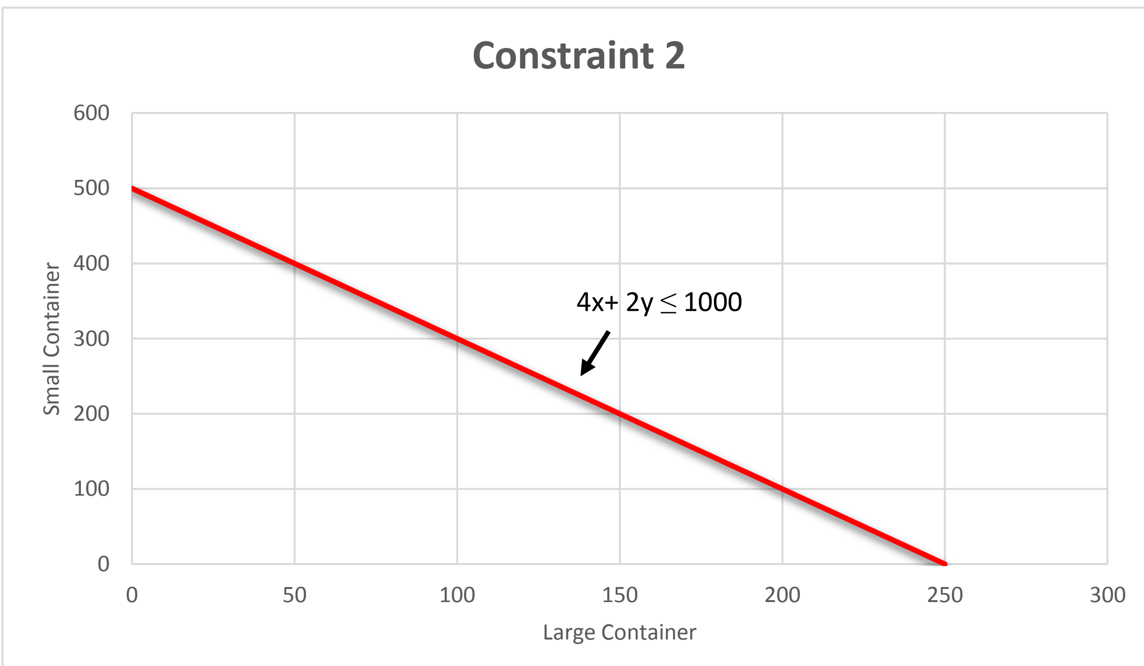
  <u>Solve for y = 0</u>

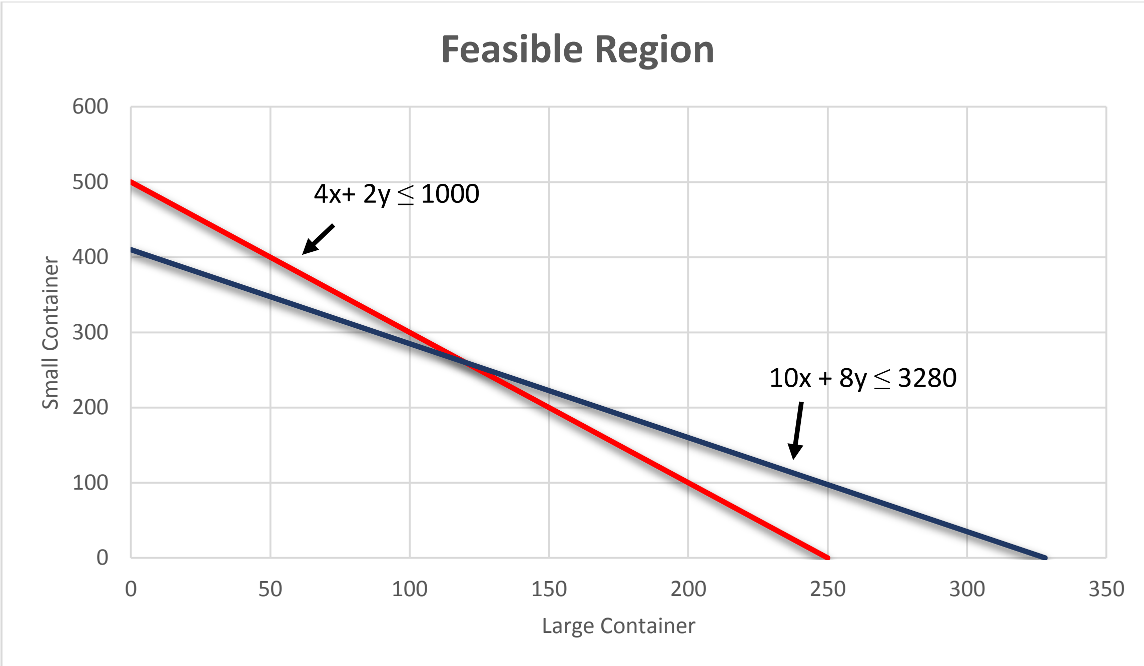
  $4x + 2(0) = 1000$

  $4x + 0 = 1000$

  $x = 1000/4$

  $x = 250$





Calculating the Optimal Solution:

Constraint 1 - $10x + 8y \leq 3280$ (Dark Blue Line above)

Constraint 2 - $4x + 2y \leq 1000$ (Red Line above)

Solve for x:

$10x + 8y = 3280$

$10x = 3280 - 8y$

$x = 328 - .8y$

Substitute x into Constraint 2:

$4(328 - .8y) + 2y = 1000$

$1312 - 3.2y + 2y = 1000$

$-1.2y = -312$

$y = 260$

Substitute y into Constraint 1 solved for x:

328 - .8(260) = x

328 – 208 = x

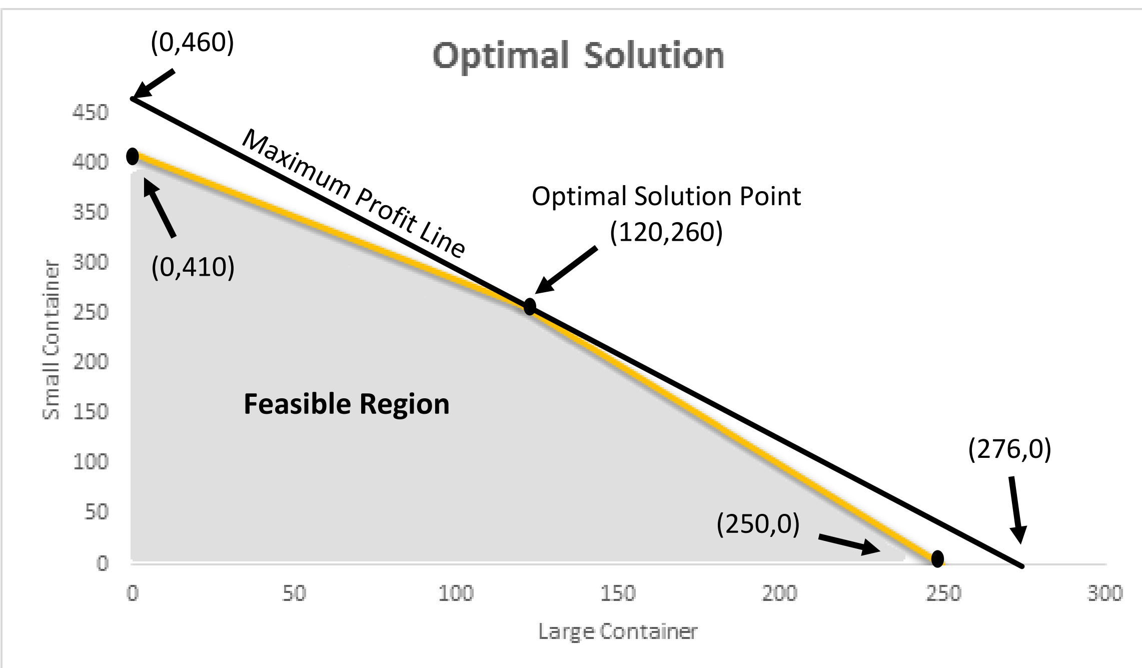
x = 120

Therefore, the coordinates for the highest profit yielding corner (optimal solution) are x = 120, y = 260; this means that the optimal hauling ratio is 120 large containers and 260 small containers yielding an optimal profit of $138:

.50x + .30y
.50 (120) + .30 (260)

60 + 78 = $138

Graphical Representation of the Optimal Solution:



Evaluate and Compare Corners:

To prove that (120,260) is the optimal corner the other two corners, (0,460) and (276,0), must be evaluated.

Corner 1: (0,410)

.50 (0) + .30 (410)

0 + 123 = $123; Which is less than $138, not the optimal corner

Corner 2: (250,0)

.50 (250) + .30 (0)

125 + 0 = $125; Which is less than $138, not the optimal corner

Evaluation:

With the above calculations as proof the optimal solution point has been defined as (120,260); yielding the highest profit of $138.

Question 2:

Now write the algorithmic statement (equations) for this problem. Input them in Lingo, run and verify that the solution is the same as above. Show your Lingo problem and results, including the analysis of ranges.

Describe the Objective:
Maximize total profit of hauling crates x (large) and y (small)
Max .50x + .30y

Describe each Constraint:
Constraint 1: $10x + 8y \leq 3280$ (Maximum weight must be LESS THAN 3280 lb)
Constraint 2: $4x + 2y \leq 1000$ (Maximum space must be LESS THAN 1000 cu-ft)

Nonnegativity Constraints:
$x, y \geq 0$ (Cannot haul a negative amount of containers)

Full Mathematical Model:

```
MAX = .50*x + .30*y;
!Subject to;
10*x + 8*y <= 3280;
4*x + 2*y <= 1000;
x>=0;
y>=0;
END
```

Lingo Solver Results:

```
Global optimal solution found.
Objective value:                              138.0000
Infeasibilities:                              0.000000
Total solver iterations:                             2
Elapsed runtime seconds:                          0.08

Model Class:                                        LP

Total variables:                    2
Nonlinear variables:                0
Integer variables:                  0

Total constraints:                  5
Nonlinear constraints:              0

Total nonzeros:                     8
Nonlinear nonzeros:                 0


                    Variable           Value        Reduced Cost
                           X        120.0000            0.000000
                           Y        260.0000            0.000000

                         Row    Slack or Surplus      Dual Price
                           1        138.0000            1.000000
                           2        0.000000            0.1666667E-01
                           3        0.000000            0.8333333E-01
                           4        120.0000            0.000000
                           5        260.0000            0.000000
```

- The above solution calculated using the Lingo software agrees with the manual solution calculation using equations and graphs in an objective value of $138 in profit through hauling 120 large containers and 260 small containers, calculated below;
    - .50x + .30y
    - .50 (120) + .30 (260)
    - 60 + 78 = $138

- This solution leaves no slack or surplus values; meaning that the optimal solution puts the trucks at full hauling capacity considering both weight limits and space limits, calculated below;
  - Large container weight (10lbs * 120 = 1200 lbs) + Small container weight (8lbs * 260 = 2080) 1200 + 2080 = 3280 lbs which is equal to the maximum hauling load of the trucks
  - Large container space (4 cu ft * 120 = 480) + Small container weight (2 cu ft * 260 = 520) 480 + 520 = 1000 cu ft which is equal to the maximum allowed space within the trucks

```
 Ranges in which the basis is unchanged:

                          Objective Coefficient Ranges:

                          Current          Allowable        Allowable
            Variable      Coefficient      Increase         Decrease
                 X        0.5000000        0.1000000        0.1250000
                 Y        0.3000000        0.1000000        0.5000000E-01


                         Righthand Side Ranges:

                     Current          Allowable        Allowable
          Row        RHS              Increase         Decrease
           2         3280.000         720.0000         780.0000
           3         1000.000         312.0000         180.0000
           4         0.000000         120.0000         INFINITY
           5         0.000000         260.0000         INFINITY
```

Question 3:

Find, by solving the equations above by hand, the Range of Optimality for the coefficient of the first variable (x), given that the coefficient of the second (y) remains at 30. Verify the Range coincides with that obtained with Lingo. Interpret this result in your problem.

- To calculate the Range of Optimality, for the coefficient x, the slope of the two binding constraint lines, Constraint 1 ($10x + 8y \leq 3280$) and Constraint 2 ($4x + 2y \leq 1000$), must first be calculated.
  - **Constraint 1**
  - $10x + 8y = 3280$
  - $8y = -10x + 3280$
  - $y = -10x/8 + 3280$
  - Slope = -5/4

- **Constraint 2**
- $4x + 2y = 1000$
- $2y = -4x + 1000$
- $y = -4x/2 + 1000$
- Slope = -2

- Now, the range of values where the objective function line is between the two binding constraint lines is called the range of optimality. Therefore, the following calculation is required:
  - $-5/4 \geq x/.3 \geq -2/1$
  - $5/4 \leq x/.3 \leq 2/1$ (Multiply by -1)
  - $.375 \leq x \leq .6$ (Multiply by .3)
- Therefore, the range of optimality is $.375 \leq x \leq .6$; this means if the coefficient of x (.50) remains in between the two values the optimal solution will not change.
  - The manual calculation has deciphered the maximum increase to be .6 - .5 = .1 and the maximum decrease to be .5 - .375 = .125
  - Manual calculation increase (.1) = Lingo calculation increase (`0.1000000`)
  - Manual calculation decrease (.125) = Lingo calculation decrease (`0.1250000`)
- As described above the results per the range of optimality values from the displayed manual calculations match the results provided through the Lingo software calculations.

Question 4:

Find, in the Lingo results, the Shadow (Dual) Price of an extra unit (one pound) of weight (constraint one). Obtain from the Lingo results the Range of Feasibility for the Dual Price and interpret these results in the context of your problem.

- The Shadow (Dual) Price is defined as the change within the objective function value per unit of increase or decrease to the right hand side of a constraint
- The Shadow (Dual) Price for constraint one calculated through the Lingo software is as follows:

```
Row    Slack or Surplus      Dual Price
2        0.000000            0.1666667E-01
```

- This .0167 is calculated below:
    - 10x + 8y = 3300 (20 lbs is added to constraint 1 to calculate the resulting change)
    - x = 330 - .8y (Solve for x)
    - 4 (330 - .8y) + 2y = 1000 (Plug into constraint 2)
    - 1320 – 3.2y + 2y = 1000
    - -1.2y = -320
    - **y = 266.6666667**

    - 330 - .8 (266.6666667) = x (Plug into "Solve for x" above)
    - 330 – 213.3333333 = x
    - **116.6666667 = x**

    - .50 (116.6666667) + .30 (266.6666667) (Plug into the objective function)
    - 58.33333335 + 80.00000001 = **138.3333334**
    - 138.3333334 – 138 (Subtract from original profit to determine total increase)
    - .3333334/20 = .01666667 (Divide by 20 lbs increase to see incremental change)
- The above calculated amount .01666667 reconciles with the Lingo calculated amount of `0.1666667E-01`
- This amount is the Shadow or Dual price and represents the change in total profit by adding or decreasing 1 lbs to constraint 1. For instance, if the trucks were able to haul 3281 lbs instead of 3280 the profit would increase by the shadow price from $138 to $138.016666667. This same rule applies for deductions to hauling capacity as if the capacity reduced to 3279 the resulting profit would decrease .016666667 from 138 to 137.8333333.
- This shadow price is only valid within a certain range of values for constraint 1; The Range of Feasibility.
- The Range of Feasibility can be calculated through the information provided within the Lingo solution, the below section is specific to constraint 1:

Row	Current RHS	Allowable Increase	Allowable Decrease
2	3280.000	720.0000	780.0000

- The allowable decrease of 780 and the allowable increase of 720 are respective to the RHS (right hand side) of constraint 1, 3280. Therefore, the Range of Feasibility can be calculated as follows:
  - 3280 – 780 = 2500 as the minimum
  - 3280 + 720 = 4000 as the maximum
  - This offers the conclusion that the .016666667 shadow price is valid through the Range of Feasibility between 2500 lbs and 4000 lbs for constraint 1.

Question 5:

Assume that now there is a third type of container available: 6 cu-ft. and weighing 12 lb. The company will charge 80 cents for each one. Redo the LP analysis and select the best of the two options. Justify your selection.

- The addition of a third container will require adjustments to both the objective function and related constraints, which will result in changes to all other calculations
- Therefore, the new model is as follows:

Describe the Objective:
Maximize total profit of hauling crates x (large), y (small), z (larger)
Max .50x + .30y + .80z

Describe each Constraint:
Constraint 1: $10x + 8y + 12z \leq 3280$ (Maximum weight must be LESS THAN 3280 lb)
Constraint 2: $4x + 2y + 6z \leq 1000$ (Maximum space must be LESS THAN 1000 cu-ft)

Nonnegativity Constraints:
$x, y, z \geq 0$ (Cannot haul a negative amount of containers)

Full Mathematical Model:
```
MAX = .50*x + .30*y + .80*z;
!Subject to;
10*x + 8*y + 12*z <= 3280;
4*x + 2*y + 6*z<= 1000;
x>=0;
y>=0;
z>=0;
END
```

Lingo Solver Results:

```
  Global optimal solution found.
  Objective value:                              144.0000
  Infeasibilities:                              0.000000
  Total solver iterations:                             2
  Elapsed runtime seconds:                          0.04


  Model Class:                                        LP

  Total variables:                      3
  Nonlinear variables:                  0
  Integer variables:                    0

  Total constraints:                    5
  Nonlinear constraints:                0

  Total nonzeros:                      11
  Nonlinear nonzeros:                   0


                                Variable           Value        Reduced Cost
                                       X        0.000000        0.5000000E-01
                                       Y        320.0000        0.000000
                                       Z        60.00000        0.000000

                                     Row    Slack or Surplus      Dual Price
                                       1        144.0000        1.000000
                                       2        0.000000        0.8333333E-02
                                       3        0.000000        0.1166667
                                       4        0.000000        0.000000
                                       5        320.0000        0.000000
                                       6        60.00000        0.000000
```

Analysis of Ranges:

```
Ranges in which the basis is unchanged:

                   Objective Coefficient Ranges:

                  Current           Allowable        Allowable
  Variable        Coefficient       Increase         Decrease
         X        0.5000000         0.5000000E-01    INFINITY
         Y        0.3000000         0.2333333        0.3333333E-01
         Z        0.8000000         0.1000000        0.1000000

                       Righthand Side Ranges:
                    Current           Allowable        Allowable
        Row         RHS               Increase         Decrease
          2         3280.000          720.0000         1280.000
          3         1000.000          640.0000         180.0000
          4         0.000000          0.000000         INFINITY
          5         0.000000          320.0000         INFINITY
          6         0.000000          60.00000         INFINITY
```

Before and After Comparison:

- Profit before inclusion of container z = $138
- Profit after inclusion of container z = $144
- These values give the clear advantage to the inclusion of container z, as profit would increase by $144 - $138 = $6
- Similar to the previous problem the optimal solution to this problem leaves no slack to the 2 variables that are utilized; y and z, this is calculated below:
    - Larger container weight (12lbs * 60 = 720 lbs) + Small container weight (8lbs * 320 = 2,560) 720 + 2,560 = 3280 lbs which is equal to the maximum hauling load of the trucks
    - Larger container space (6 cu ft * 60 = 360) + Small container weight (2 cu ft * 320 = 640) 360 + 640 = 1000 cu ft which is equal to the maximum allowed space within the trucks
- This second scenario also has a larger range of optimality and range of feasibility for both utilized variables:
    - Scenario 1:

Ranges in which the basis is unchanged:

Objective Coefficient Ranges:

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
X	0.5000000	**0.1000000**	**0.1250000**
Y	0.3000000	**0.1000000**	**0.5000000E-01**

Righthand Side Ranges:

Row	Current RHS	Allowable Increase	Allowable Decrease
2	3280.000	**720.0000**	**780.0000**
3	1000.000	**312.0000**	**180.0000**

- Scenario 2:

Ranges in which the basis is unchanged:

Objective Coefficient Ranges:

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
Y	0.3000000	**0.2333333**	**0.3333333E-01**
Z	0.8000000	**0.1000000**	**0.1000000**

Righthand Side Ranges:

Row	Current RHS	Allowable Increase	Allowable Decrease
2	3280.000	**720.0000**	**1280.000**
3	1000.000	**640.0000**	**180.0000**

- By looking at the bolded numbers above one can see the larger area within both the Range of Optimality and the Range of Feasibility lies with scenario 2; this is a good situation for the hauling company as it allows for a greater amount of fluctuation without a change in the optimal solution or shadow price
- Therefore, my recommendation to the company would be to begin using container z within their hauling company as profits would be higher by $6, in addition the optimal solution for this scenario puts the number of “large” containers at 0 allowing the company to potentially abandon this type of container. The larger Range of Optimality and Range of Feasibility also offer another advantage to the second scenario making the optimal solution and shadow price less susceptible to change.