START Selected Topics in Assurance Related Technologies

Volume 10, Number 3

Empirical Assessment of Weibull Distribution

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Introduction

This START sheet discusses some empirical and practical methods for checking and verifying the statistical assumptions of the Weibull distribution. It presents several numerical and graphical examples and provides references for further reading.

It is important to correctly assess statistical distributions. For, when our hypothesized distribution does not hold, the derived statistical results are invalid (6). For example, the confidence levels of the confidence intervals (or of hypotheses tests) implemented may be completely off. To avoid such problems, we need to check all distribution assumptions.

Two approaches are used to assess the distribution assumptions. One is by implementing numerically convoluted, theoretical Goodness of Fit (GoF) tests such as the Chi Square, Anderson Darling or Kolmogorov-Smirnov. Their lengthy calculations often require the use of specialized software, not always readily available. On the other hand, there exist practical procedures that are easy to understand and implement and are based on intuition and graphical distribution properties. These procedures can also be used to assess the distribution assumptions (5, 7, 8).

This START sheet discusses such practical assessment procedures, for the important case of the Weibull distribution, widely used in reliability, maintainability, and safety (RMS) work (1, 2, 3, 4). We begin with a numerical example that illustrates the importance of this problem. Then, we develop additional numerical and graphical examples that illustrate the implementation and interpretation of such distribution checks.

Putting the Problem in Perspective

Assume that we need to estimate the reliability of a device, R(T), for a Mission Time T, based on some life data (X_1, ..., X_n). First, consider that the distribution of the life of a device (times to failure) is Weibull (Figure 1) and then that it is Exponential (Figure 2), having the same mean = 10. Figures 1 and 2 were obtained from 5000 data points from each of these two distributions. The Weibull, in addition, has shape parameter β = 1.23 and scale parameter α = 11.

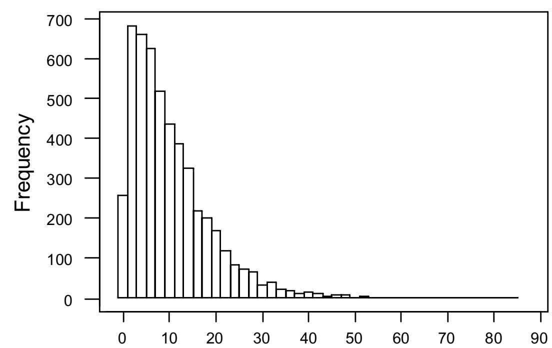

Figure 1. Weibull (α = 11, β = 1.23)

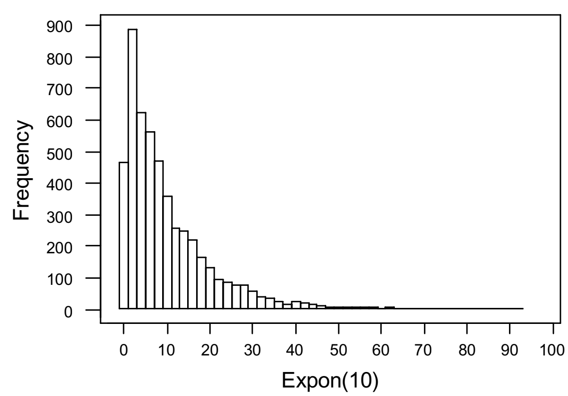

Figure 2. Exponential (θ = 10)

The descriptive statistics for these 5000 data points are shown in Table 1. Notice how the two means are 10. The two distributions differ mainly in that Weibull clusters about the mean and is therefore, less variable than the Exponential (contrast the StDev values).

Table 1. Descriptive Statistics for the Data Sets

Variable	N	Mean	Median	StDev	Min	Max	Q1	Q3
W(11,1.23)	5000	10.106	7.936	8.338	0.010	77.834	3.875	14.010
Expon(10)	5000	9.996	6.868	10.174	0.001	92.951	2.736	13.933

There are some practical connotations of belonging to one of these two distributions. The Weibull distribution with shape parameter larger than unity ($\beta > 1$) characterizes a life that deteriorates with time, i.e., device lives whose failure rate increases with time (reliability decay). On the other hand, when the shape parameter is unity ($\beta = 1$), Weibull becomes an Exponential distribution. Hence, the device failure rate is constant and there is no reliability growth or decay. Finally, if the shape parameter is smaller than unity ($\beta < 1$), there is reliability growth because the failure rate of the device decreases with time.

Thus, a point estimator based on the life data is obtained by calculating such reliability according to some "formula." However, reliability is defined as the probability that a device life X outlasts the device mission time T (formally, $R(T) = P\{X > T\}$). As a result, the assumption of a specific statistical distribution for the device life determines which "formula" we use, as well as which parameters it includes.

For example, assume the data are distributed as a Weibull, with shape parameter β and scale parameter α. Then, the "formula" of the Weibull reliability point estimator is:

$$R(T) = P\{X \geq T\} = \mathrm{Exp}\{-(T/\alpha)^{\beta}\}$$

However, if the data are assumed Exponential, with mean θ, the Exponential reliability estimator becomes:

$$R(T) = P\{X \geq T\} = \mathrm{Exp}\{-T/\theta\}$$

Because the two distributions are different the two reliability estimations will differ (they have different formulas and parameters) except when the shape parameter $\beta = 1$ and the Weibull distribution becomes an Exponential.

For example, if the required Mission Time is T = 3 and the parameters are known and equal to $\alpha = 11$, $\beta = 1.23$ and $\theta = 10$, the two respective reliabilities are as follows:

If the true distribution of lives were Weibull (11,1.23):

$$R(T) = \mathrm{Exp}\{-(T/\alpha)^{\beta}\} = \mathrm{Exp}\{-(3/11)^{1.23}\} = 0.81$$

If the true distribution of lives were Exponential (10):

$$R(T) = \mathrm{Exp}\{-T/\theta\} = \mathrm{Exp}(-3/10) = 0.74$$

The difference between the two reliabilities is close to 10%! Thus, it is very importance to assess (via the sample data) whether or not that our distribution assumption is correct.

Finally, the problem becomes yet more complex when the distribution parameters are unknown. For then we also need to estimate these parameters from the samples and the uncertainty increases even more.

Statistical Assumptions and their Implications

Fortunately, distribution model assumptions are associated with very practical and useful implications, and the Weibull is no exception. In practice, the assumption that Weibull is the true distribution of the lives of a device has several important connotations: some physical and theoretical and others algebraic and graphical.

The physical interpretations can be inferred from Weibull's relationship to the Extreme Value Theory (3, 4). For example, consider a metallic chain where each of its "n" links has the same size and strength. Such a chain can be considered a series system composed of "n" components, each having the same life distribution and failure rate. The system fails whenever the first failure occurs (link breaks). Therefore, the lives of a population of these systems (chains) would follow the Weibull.

In addition, the Weibull failure rate increases, decreases or remains constant, according to the value of shape parameter β. These characteristics help us assess whether the life of a device is Weibull, by analyzing its physical conditions.

The algebraic consequences stem from another important characteristic of the Weibull: its closed functional forms that are easily manipulated from a mathematical standpoint. Weibull's density and distribution functions are, respectively:

$$f(x) = \frac{\beta}{\alpha^{\beta}} x^{\beta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\} \text{ and } F(x) = 1 - \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}$$

The graphical consequences stem from such ease of algebraic manipulations. Taking the logarithms of the distribution function F(x) and doing some algebra, we obtain:

$$\ln\left\{\ln\left(\frac{1}{1 - F(x)}\right)\right\} = -\beta\ln(\alpha) + \beta\ln(x)$$

When the distribution of the lives is really Weibull, the previous equation is that of a line. Now assume that an estimation of F(x) can be obtained and denote it p_x. We then can substitute p_x in lieu of F(x) in the equation and solve for x.

We actually estimate the value p_x for any data point x, i.e., the "median rank" by defining:

$$F(x) \cong p_x = \frac{\mathrm{Rank}(x) - 0.3}{n + 0.4}$$

where Rank(x) is the rank of life x, in the sorted sample of size n, of all device lives.

Using such p_x values, we plot the pairs (p_x, x) in "Weibull paper". Alternatively, we can plot the Log-transformed, sorted data, right from the above equation, as will be shown in the next section. In either case, we use these plots to assess whether the true distribution is Weibull, and to estimate its parameters.

Practical Methods to Verify Weibull Distribution

We now apply several empirical and practical procedures to the life test data in Table 2 to determine if the sample (n = 45) was taken from the Weibull.

Table 2. Large Sample Life Data Set (sorted)

0.8997	1.2838	1.5766	1.8627	2.4193	2.4353
3.1520	3.3367	3.4850	3.9605	3.9921	3.9934
4.1013	4.8306	5.3545	5.6094	7.7829	7.8240
8.3431	9.0248	9.2627	9.2766	9.7943	11.4391
12.2847	12.4112	13.1651	13.4990	13.5532	14.1542
14.4694	14.5857	15.1603	15.6962	15.7833	17.4998
18.1497	18.6342	19.4354	19.7557	19.9496	22.5383
23.8066	29.9006	34.0658			

In this life data set, two distribution assumptions need to be verified: (1) that the data are independent and (2) that they are identically distributed as a Weibull.

The assumption of independence implies that randomization (sampling) of the population of devices (and other influencing factors) must be performed before placing them on test. For example, device operators, times of operations, weather conditions, location of the devices in warehouses, etc. should be randomly selected. Only then will the sample be representative of the population.

To assess the second assertion, we use informal methods, based on the properties of the Weibull distribution. They seem appropriate for the practical engineer, since they are largely intuitive and easy to implement.

To assess a sample, we first tabulate and plot the raw data in several ways. The descriptive statistics are shown in Table 3 and the histogram in Figure 3. Next, we analyze and check (empirically but efficiently) if a Weibull assumption holds.

Table 3. Descriptive Statistics of Data in Table 2

N	Mean	Median	StdDev	Min	Max	Q1	Q3
45	11.19	9.79	7.85	0.9	34.07	3.99	15.74

There are a number of useful and easy to implement procedures, based on well-known statistical properties of the Weibull distribution, which help us to informally assess this assumption. These properties are summarized in Table 4.

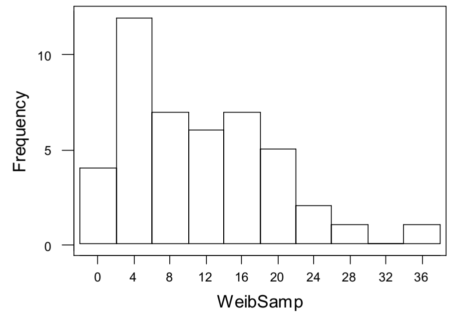

Figure 3. Histogram of the Sample from Table 2

Table 4. Some Properties of the Weibull Distribution

1. Characteristic life α lies approximately at the 63^{rd} percentile (63% of the population). Hence, the Weibull sample should replicate this. Sample 63^{rd} percentile should be an alternative (gross) estimator of characteristic life α.
2. The plot of the transformed, sorted data set of lives $\{X_1, ..., X_n\}$:

$$\ln\left\{\ln\left(\frac{1}{1-p_X}\right)\right\} \text{ versus } \ln(x) \text{ where } p_X = \frac{\text{Rank}(x) - 0.3}{n+0.4}$$

should be linear, if the true distribution is Weibull.
3. The slope of the linear trend from Property 2 is an alternative estimator of shape β.
4. The regression of the pairs defined in Property 2, yields better estimates of (α, β) and these should be close to the raw estimates obtained in Properties 2 and 3 above.
5. The transformation $Y = X^{\beta}$ should yield an Exponential distribution with mean $\mu = \alpha^{\beta}$.
6. The Weibull Probability and Score plots of device lives $\{X_1, ..., X_n\}$ should be linear.
7. The corresponding regressions from the plots in Property 6 should have a slope of unity.

To verify Property 1, we notice how device lives 13.49 and 13.55 (in Table 2) have ranks 28 and 29. Since the 63^{rd} percentile is estimated by 0.63*n = 0.63*45 = 28.35 we need to interpolate. The average of these two lives (13.53) yields a rough estimate of the Weibull characteristic life α, which we will compare with results from Properties 4 and 7.

To verify Property 2, we transform the data (Table 5). The first column is the original data, the second its mean rank p_x, the third its transformation $\ln(\ln(1/(1- p_x)))$ and the last column, its transformation ln(X).

For example, for the first (smallest) value (0.8997) p_x is:

$$P_X = \frac{\text{Rank}(x) - 0.3}{n+0.4} = \frac{1-0.3}{45+0.4} = \frac{0.7}{45.4} = 0.0154$$

Substituting p_x for F(X) in $\ln(\ln(1/(1- F(X))))$ we obtain the corresponding:

$$\ln\left\{\ln\left(\frac{1}{1-p_X}\right)\right\} = \ln\left\{\ln\left(\frac{1}{1-0.0154}\right)\right\} = \ln\{\ln(1.0156)\}$$

$$= \ln(0.0155) = -4.165$$

Table 5. Transformed Data

Row	Sample	Px	Ln(Ln(*)	Ln(X)
1	0.8997	0.0154	-4.1644	-0.10566
2	1.2838	0.0374	-3.2659	0.24980
3	1.5766	0.0595	-2.7918	0.45529
4	1.8627	0.0815	-2.4650	0.62200
5	2.4193	0.1035	-2.2138	0.88347
6	2.4353	0.1256	-2.0087	0.89009
7	3.1520	0.1476	-1.8346	1.14804
8	3.3367	0.1696	-1.6828	1.20498
9	3.4850	0.1916	-1.5477	1.24846
10	3.9605	0.2137	-1.4256	1.37636
11	3.9921	0.2357	-1.3139	1.38432
12	3.9934	0.2577	-1.2106	1.38465
13	4.1013	0.2797	-1.1143	1.41131
14	4.8306	0.3018	-1.0239	1.57496
15	5.3545	0.3238	-0.9384	1.67794
16	5.6094	0.3458	-0.8572	1.72444
17	7.7829	0.3678	-0.7795	2.05193
18	7.8240	0.3899	-0.7051	2.05720
19	8.3431	0.4119	-0.6333	2.12143
20	9.0248	0.4339	-0.5638	2.19998
21	9.2627	0.4559	-0.4964	2.22599
22	9.2766	0.4780	-0.4307	2.22750
23	9.7943	0.5000	-0.3665	2.28180
24	11.4391	0.5220	-0.3035	2.43704
25	12.2847	0.5441	-0.2416	2.50836
26	12.4112	0.5661	-0.1805	2.51860
27	13.1651	0.5881	-0.1199	2.57757
28	13.4990	0.6101	-0.0598	2.60262
29	13.5532	0.6322	0.0001	2.60663
30	14.1542	0.6542	0.0600	2.65001
31	14.4694	0.6762	0.1201	2.67204
32	14.5857	0.6982	0.1808	2.68004
33	15.1603	0.7203	0.2421	2.71868
34	15.6962	0.7423	0.3045	2.75342
35	15.7833	0.7643	0.3683	2.75895
36	17.4998	0.7863	0.4340	2.86219
37	18.1497	0.8084	0.5021	2.89866
38	18.6342	0.8304	0.5734	2.92500
39	19.4354	0.8524	0.6489	2.96710
40	19.7557	0.8744	0.7300	2.98344
41	19.9496	0.8965	0.8189	2.99321
42	22.5383	0.9185	0.9192	3.11522
43	23.8066	0.9405	1.0375	3.16996
44	29.9006	0.9626	1.1893	3.39788
45	34.0658	0.9846	1.4284	3.52830

We plot the pairs [ln{ln(1/1 - p_x)}, ln(x)] in Figure 4. They reflect a linear trend, as expected from Property 2, when the device lives are distributed as a Weibull.

From Figure 4 and the data in Table 5, we obtain the slope for the estimated linear trend:

$$\text{Slope} = \frac{F(2.65) - F(2.19)}{2.65 - 2.19} = \frac{0.06 - (-0.56)}{2.65 - 2.19} = \frac{0.62}{0.46} = 1.3478$$

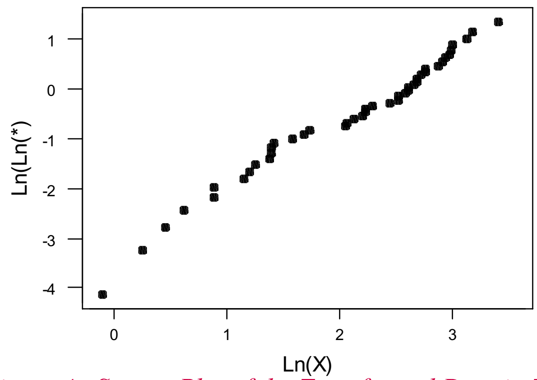

Figure 4. Scatter Plot of the Transformed Data in Table 5 (in Columns 2 and 4)

This slope (1.3478) is a rough estimate of the Weibull shape parameter β. To obtain a formal estimation (Property 4, of Table 4) we regress ln(ln(1/(1- p_x))) = C2 and ln(X) = C1:

The regression equation is C2 = -3.41 + 1.35 C1

Predictor	Coef	StDev	T	P
Constant	-3.40715	0.06856	-49.69	0.000
C1	1.35424	0.03008	45.02	0.000

S = 0.1774 R-Sq = 97.9% R-Sq(adj) = 97.9%

Intercept = -3.4071, Slope = 1.3542

The regression fit is high (97.9%); its slope (1.35) is the Weibull shape parameter; CharLf is the Weibull Characteristic Life, or scale parameter, and it is obtained by:

CharLf = Exp(-(Intercept/Slope)) = Exp(-(-3.4/1.35)) = 12.378

Notice how the rough estimates of Characteristic Life and shape parameters (13.53 and 1.347) are close to the more formal Weibull estimates given by the regression above.

We now perform the transformation $Y = X^{\beta}$ (Table 6). If X is distributed Weibull then, by Property 5 in Table 4, Y will be Exponential with mean α^{β}.

The Exponentiality of Y can be assessed by any or all of the procedures in Reference 5. For example, compare the descriptive statistics and probability plots of variable $Y = X^{\beta}$.

Table 6. Transformation Y = X**1.35 Yields an Exponential (μ = 29.860)

0.867	1.401	1.849	2.316	3.296	3.325	4.711
5.087	5.395	6.411	6.481	6.484	6.721	8.383
9.633	10.257	15.960	16.074	17.530	19.491	20.188
20.229	21.768	26.843	29.556	29.967	32.450	33.567
33.749	35.785	36.865	37.265	39.261	41.146	41.454
47.654	50.058	51.870	54.904	56.129	56.874	67.057
72.201	98.213	117.120				

Notice in Table 7 how the mean and standard deviation of $Y = X^\beta$ are relatively close, as expected in an Exponential distribution. The Probability plot of Y, presented in Figure 5, also shows a clear linear trend.

Table 7. Descriptive Statistics

Variable	N	Mean	Median	StDev
Transf	45	28.97	21.77	26.11

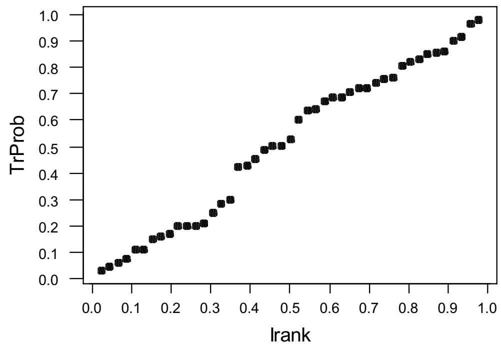

Figure 5. Probability Plot for the Transformed Variable $Y = X^\beta$ (Linear Trend as Expected for Exponential)

To assess Property 6 in Table 4, we implement Weibull probability and score plots on the original lives $\{X_1, \ldots, X_n\}$. These plots (Figures 6 and 7) as expected, are linear.

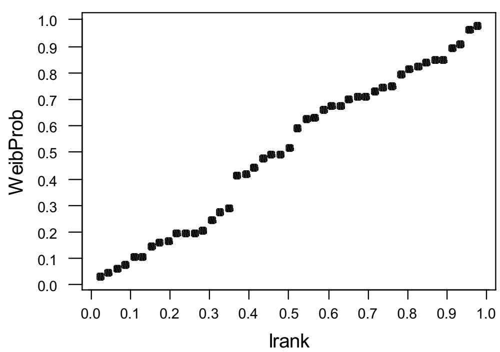

Figure 6. Probability Plot for the Weibull Data; it Follows an Upward Linear Trend, as Expected if X is Weibull

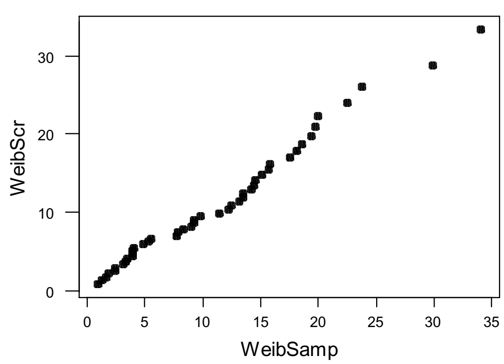

Figure 7. Weibull Scores Plot Displays a Linear Trend, as Expected from Property 7

If the Weibull assumption is correct, the linear regression of the data in Figure 6 should also reflect the one-to-one relation, yielding a slope of unity (Property 7).

The regression equation is WeibProb = -0.0113 + 1.03 Irank

Predictor	Coef	StDev	T	P
Constant	-0.01131	0.01177	-0.96	0.342
Irank	1.03051	0.02049	50.29	0.000

S = 0.03881 R-Sq = 98.3% R-Sq(adj) = 98.3%

The regression Index of Fit is very high ($R^2 = 98.3$). The regression slope (1.03) yields an approximate 99% CI (0.97, 1.09) that covers unit, supporting Weibull by Property 7.

The Weibull scores (x_i) are the percentiles corresponding to the Median Ranks p_x in Table 5. To obtain such percentiles, we substitute p_x for F(x) in the Weibull equation

$$F(x) = 1 - \exp\left\{ - \left(\frac{x_i}{\alpha} \right)^\beta \right\},$$

and solve for X_i obtaining the equation

$$x_i = \sqrt[\beta]{-\alpha^\beta \ln(1 - p_x)}$$

For example, from the smallest data point (0.899) we get the first Weibull score (using $p_x = 0.0154$) in the following manner:

$$x_1 = \sqrt[1.35]{-(12.38)^{1.35} \ln(1 - 0.0154)} = 0.566$$

Weibull scores are then plotted vs. their corresponding sorted data (e.g., 0.566 vs. 0.899). The Weibull scores plot is presented in Figure 7.

The regression of the Weibull scores on the ordered sample, according to Property 7 in Table 4, should also yield a slope of unity. The regression equation is:

WeibScr = 0.041 + 0.989 WeibSamp

Predictor	Coef	StDev	T	P
Constant	0.04080	0.26870	0.15	0.880
WeibSamp	0.98886	0.01974	50.11	0.000

S = 1.027 R-Sq = 98.3% R-Sq(adj) = 98.3%

As with the Probability Plot, the Index of Fit (98.3%) is very high. The 99% approximate CI (0.92, 0.104) also covers unity, as expected when the data is distributed Weibull.

All of the preceding empirical results support the plausibility of the Weibull assumption for our life data set. If, at such point, a stronger case for the validity of the Weibull distribution is required, then a number of theoretical GoF tests can be carried out. GoF tests will be the topic of a forthcoming paper.

Summary

In this START sheet we have discussed the important problem of (empirically) assessing the Weibull distribution assumptions of a data set. We have provided several numerical and graphical examples. We have discussed some related theoretical and practical issues, giving references to background information and further readings. In doing so, we mentioned other, very important, reliability analysis topics. Due to their complexity, these will be treated in more detail in forthcoming papers.

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Dr. Jorge Luis Romeu has over thirty years of statistical and operations research experience in consulting, research, and teaching. He was a consultant for the petrochemical, construction, and agricultural industries. Dr. Romeu has also worked in statistical and simulation modeling and in data analysis of software and hardware reliability, software engineering and ecological problems.

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