NOTES ON INDUCTION

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Definitions & Theorems

Axiom: (The Principle of Mathematical Induction - PMI) Let \( P(n) \) be a statement for each \( n \in \mathbb{N} \). We conclude \( P(n) \) is true for all \( n \in \mathbb{N} \) if the following two conditions are met.
(1) \( P(1) \) is true.
(2) \( \forall k \in \mathbb{N}, P(k) \Rightarrow P(k+1) \).

Axiom: (The Generalized Principle of Mathematical Induction - GPMI) For any integer \( m \), let \( S_m = \{ i \in \mathbb{Z} : i \geq m \} \) and suppose \( P(n) \) is a statement for each \( n \in S_m \). We conclude \( P(n) \) is true for all \( n \in S_m \) if the following two conditions are met.
(1) \( P(m) \) is true.
(2) \( \forall k \in S_m, P(k) \Rightarrow P(k+1) \).

Axiom: (The Strong Principle of Mathematical Induction - SPMI) For any integer \( m \), let \( S_m = \{ i \in \mathbb{Z} : i \geq m \} \) and suppose \( P(n) \) is a statement for each \( n \in S_m \). We conclude \( P(n) \) is true for all \( n \in S_m \) if the following two conditions are met.
(1) \( P(m) \) is true.
(2) \( \forall k \in S_m, (P(m) \land P(m+1) \land \ldots \land P(k)) \Rightarrow P(k+1) \).

Comment: A proof that utilizes any one of the PMIs above us usually referred to as an induction proof or a proof by induction; however, some authors are careful to distinguish between a proof by PMI and a proof by SPMI.

Definition: Verifying the truth of condition (1) in an induction proof is called the base step or basis step.

Definition: Verifying condition (2) in an induction proof is called the inductive step. In the inductive step we are verifying an implication, so we need only confirm that the conclusion follows from the assumption of the premise. We refer to the premise \( P(k) \) the inductive hypothesis and we refer to the premise \( P(m) \land P(m+1) \land \ldots \land P(k) \) as the strong inductive hypothesis.
Definition: Let $S$ be a set positive of integers. When a statement $\forall n \in S, P(n)$ is false, there is some nonempty subset $T \subseteq S$ for which $\forall n \in T, \neg P(n)$ is true. The least element of $T$ is referred to as the **minimum counterexample** to the statement $\forall n \in S, P(n)$.

Definition: A proof by contradiction of $\forall n \in S, P(n)$ that proceeds by assuming there is some minimum counterexample and then arriving at a contradiction is called a **proof by minimum counterexample** or a **least criminal proof**.

Definition: Let $A$ be a nonempty set of real numbers. A number $m \in A$ is called a **least element** of $A$ iff, for each $x \in A$, $x \geq m$.

Theorem: If a set $A$ of real numbers has a least element, then $A$ has a unique least element.

Definition: A nonempty set $S$ of real numbers is said to be **well-ordered** iff every nonempty subset of $S$ has a least element.

Theorem: (The Well-Ordering Principle - WOP) The positive integers are well-ordered.

Theorem: For each integer $m$, the set $S_m = \{i \in \mathbb{Z} \mid i \geq m\}$ is well-ordered.

Theorem: If a real interval has more than one element, then it is not well-ordered.

**Examples**

Example: (Not well-ordered sets)

The sets $\mathbb{Z}, \mathbb{Q},$ and $\mathbb{R}$ are not well-ordered due to the lack of least elements.

The interval $(0, 1)$ is not well-ordered because there is no least element.

The interval $[0, 1]$ is also not well-ordered because $(0, 1) \subseteq [0, 1]$.

Proposition: For any $n \in \mathbb{N}$, $3 + 8 + 13 + \ldots + (5n - 2) = \frac{n}{2}(5n + 1)$.

Proposition: For any $n \in \mathbb{N}$, $2 + 9 + 16 + \ldots + (7n - 5) = \frac{n}{2}(7n - 3)$.

Proposition: For every positive integer $n$, $6 \mid (n^3 - n)$.

Example: For every integer $n \geq 5$, $2^n > n^2 + 1$.

Example: For any $n \geq 2$, the product of any $n$ odd integers is odd.

Example: Let $n \geq 2$. Then, if a product $x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is even, then at least one of the integers $x_1, x_2, \ldots, x_n$ is even.
Example: For every integer \( n \geq 2 \), if \( x_1, x_2, \ldots, x_n \) are any \( n \) real numbers, then
\[
|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|.
\]

Theorem: Let \( n \geq 2 \). Then for any sets \( A_1, A_2, \ldots, A_n \),
\[
A_1 \cup A_2 \cup \ldots \cup A_n = \overline{A_1 \cap A_2 \cap \ldots \cap A_n}.
\]

Definition: A sequence \( \{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3, \ldots \) is said to be defined recursively
if some number \( k \in \mathbb{N} \) of terms \( a_1, a_2, \ldots a_k \) are described explicitly
and the rest are defined in terms of these initial values \( a_1, a_2, \ldots a_k \).
The rule that tells you how to find \( a_n \) in terms of \( a_1, a_2, \ldots, a_{n-1} \) is
called the recurrence relation.

Example: Given \( a_1 = 1 \) and \( a_n = a_{n-1} + 1 \) for \( n > 1 \), prove \( a_n = n \).

Example: Given \( a_1 = a \) and \( a_n = a_{n-1} + s \) for \( n > 1 \), prove \( a_n = a + ns \).

Example: Given \( a_0 = 1 \) and \( a_n = 2a_{n-1} \) for \( n > 0 \), prove \( a_n = 2^n \).

Example: Given \( a_0 = a \) and \( a_n = ba_{n-1} \) for \( n > 0 \), prove \( a_n = a \cdot b^n \).

Definition: The \( n \)th Fibonacci number \( f_n \) is defined by letting \( f_1 = 1 \) and \( f_2 = 1 \)
and for \( n > 2 \), \( f_n = f_{n-1} + f_{n-2} \).

Proposition: A Fibonacci number is even iff its index is divisible by 3.

Proposition: For the Fibonacci numbers defined above, \( f_n = \frac{\varphi^n - \overline{\varphi}^n}{\sqrt{5}} \) where
\[
\varphi = \frac{1 + \sqrt{5}}{2}
\]
is the golden ratio and \( \overline{\varphi} = 1 - \varphi \).

Proof: Note that \( \varphi + 1 = \varphi^2 \) and \( \overline{\varphi} + 1 = \overline{\varphi}^2 \). Then proceed with SPMI.