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Jorge Luis Romen

Radar Clutter Generation
for Non Gaussian
Investigations of a Model
Further Monte Carlo
Further Monte Carlo
Investigation of a Model
for Non Gaussian
Radar Clutter Generation

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Acknowledgements
Outline

- Conclusions
- Experimentation Results
- General Multivariate Case
- Parameter Estimation and Validation
- The Bivariate Case
- Problem Background
- Research Problem Statement
Research Problem:

Process: Symmetrize

\[ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n \]

Input:

\[ x_1, x_2, \ldots, x_n \]

\[ \tilde{x}_i = \frac{x_i + \tilde{x}_i}{2} \]

\[ \tilde{p}_i > \frac{1}{2} \]

Output:

\[ \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m \]

Transformation Distribution Through

\[ \tilde{x}^2 = x^2 \]
Spherically Invariant Random Processes:

\[ X = S \cdot Z \]

where: \( Z \sim \text{MVN}_N(0, M) \)

\( S \) is a univariate (indep.) process

let \( E(S^2) = 1 \) (by so defining the \( S \))

then \( \Sigma_X = \Sigma = M \) (Variance of \( X \))

let the quadratic form \( p = X^T \Sigma^{-1} X \)

then, the conditional pdf of \( X | S \) is:

\[
 f_{X|S}(x|s) = (2\pi)^{-\frac{N}{2}} |M|^{-\frac{1}{2}} S^{-N} \exp \left( -\frac{p}{2S^2} \right) 
\]

and \( f_{X}(x) = (2\pi)^{-\frac{N}{2}} |M|^{-\frac{1}{2}} h_N(p) \)

where:

\[
 h_N(p) = \int_0^{+\infty} S^{-N} \exp \left( -\frac{p}{2S^2} \right) f_S(s) \, ds
\]

with \( f_S(s) \) the pdf of the process \( S \)

Finally, the quadratic form \( p \) will have:

\[
 f_p(p) = \frac{1}{2^\frac{N}{2} \pi(N/2)} p^{\frac{N}{2} - 1} h_N(p)
\]

In particular, the K-Distributed SIRP \( X \) is:

\[
 f_X(x) = \frac{2b}{\Gamma(a)} \left( b x \right)^{a-1} K_{a-1}(bx)
\]

where \( a, b \) are shape and scale parameters and \( K_N \) is the \( N \)th order Modified Bessel Func. (2nd kind)

For such K-Distributed SIRP \( X \) we have:

\[
 f_S(s) = \frac{2}{\Gamma(a)} 2^a \left( bs \right)^{2a-1} \exp \left\{ \frac{-b^2 s^2}{2} \right\}
\]

\[
 h_N(p) = \frac{b^N (bvp)^{a - \frac{N}{2}}}{\Gamma(a)} 2^{a-1} K_{\frac{N}{2} - a}(bvp)
\]

* for long tailed K-Dist \( X \), \( a \) is very small.
For univariate Laplace SIRP $X$:

$$f_X(x) = \frac{1}{\lambda} \exp\left\{-\frac{|x-u|}{\lambda}\right\}; \quad \lambda > 0$$

To generate a suitable $X$ via Monte Carlo:

\[ w \sim \exp(1) \]

\[ y = \sqrt{2w} \sim \text{Rayleigh} \ (\text{with } \mathbb{E}(y) = 2) \]

\[ S = \frac{y}{\sqrt{2}} \] is suitable for M.C.

i) $\mathbb{E}(S) = 1$ and ii) $f_S(s) = 2s \exp(-s^2)$

then: $X = S \cdot Z \ (\text{with } Z \sim \mathcal{N}(0, 1))$

\[ p = x'x = x^2 \]

\[ f_N(p) = \sqrt{\pi} \exp(-12p) \]

\[ f_p(p) = \frac{1}{\sqrt{2p}} \exp(-\sqrt{2p}) \]

then, transforming: $t = \sqrt{2p} \sim \exp(1)$ we find:

* we can test the SIRP $X$:

$$f_X(x) = \sqrt{2\pi} \left| \Sigma \right|^{\frac{1}{2}} \phi(x) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2} |x|)$$

Laplace $w/\lambda = \frac{1}{\sqrt{2}}$

* the quadratic function $p$ (indirectly via $t$)

This is precisely the problem of M.C. experiments:

* for higher dimensions

* for $\alpha = 1$ (shape parameter of k-Shape)

* we are currently investigating for $0 < \alpha < 1$

* and for estimates of $\Sigma$

* $p = x'x$ serves for distribution identification
Our analysis yielded the conveniently tractable special case of the K-Distributed bivariate \((N = 2)\) SIRP \(X = s \ast Z\), with shape parameter \(\alpha = N/2 - 0.5\) and scale parameter \(b = 1\). This special SIRP has a quadratic form \(p\) with closed form pdf function \(f_p(*)\):

\[
\text{Given } h_N(p) = \frac{\sqrt{\pi}^{N/2-0.5-N/2}}{\Gamma(N/2-0.5)2^{N/2-0.5-1}} \times K_{N/2-N/2+0.5}(\sqrt{p}) = \frac{\sqrt{\pi} \exp(-\sqrt{p})}{p \Gamma(N-1/2)2^{N-3}}
\]

and \(K_{N-\alpha} = K_{0.5}(z) = K_{-0.5}(z) = \sqrt{\frac{\pi}{2}} \times \sqrt{\frac{1}{z}} \exp(-z)\)

Then \(f_p(p) = \frac{1}{2^1 \Gamma(1)} \times p^{1-1} h_2(p) = \frac{1}{2} p^{-1/2} \exp(-\sqrt{p})\)

These functions will provide the theoretical comparison values to validate our Monte Carlo study of the SIRP process \(X = s \ast Z\). However, in the above form, this pdf is still too complex for us to work with, directly. And certain transformations are required to simplify our work.

Under the transformation \(\frac{w^2}{2} = \sqrt{p}\), the resulting random variable (r.v.) \(w\) is distributed Rayleigh and easy to test for GOF. Such GOF test is necessary (i) to validate our Monte Carlo experiment and (ii) to compare the Power of our modified \(Q_n\) test with an established GOF test for \(p\). It is statistically equivalent to test GOF directly on the r.v. \(p\) or on its transformation, the r.v. \(w\). For, if \(p \sim F_p\) then \(w \sim \text{Rayleigh}\). We select the second alternative for its ease and computational speed.

Accordingly, the pdf \(f_S(*)\) of the driver random variable \(s > 0\), for this special case of K-Distributed SIRP \(X = s \ast Z\) is:

\[
f_S(s) = \frac{2}{\Gamma(\alpha)2^{-\alpha}2^{\alpha-1}} \exp\left(-\frac{s^2}{2}\right) = \sqrt{\frac{2}{\pi}} \exp(-s^2/2)
\]

which, under the transformation \(y = s^2/2\) becomes:

\[
f_Y(y) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} \exp(-y) = \frac{1}{\Gamma(1/2)} y^{-1/2} \exp(-y)
\]

i.e., a Gamma distribution with parameters \(\lambda = 1, r = 1/2\). It is now easier and faster to generate a Gamma r.v. \(y\) and obtain \(s = \sqrt{2y}\).

The above random variable \(s\) also has the convenient property that:

\[
E(s^2) = \int_0^\infty s^2 \sqrt{\frac{2}{\pi}} \exp(-s^2/2) \, ds = 1
\]

therefore, insuring that the resulting covariances \((\Sigma = M)\) of our Gaussian \((Z)\) and SIRP \(X = s \ast Z\) processes are equal.

We also investigated other special cases of SIRP for \(\alpha = N/2 - 0.5\) and \(b = 1\):

For \(N = 3\)

\[
f_p(p) = \frac{\sqrt{\pi}}{4 \Gamma(1.5)} e^{-\sqrt{p}}
\]

For \(N = 4\)

\[
f_p(p) = \frac{\sqrt{p^2 \pi}}{8 \Gamma(2)} e^{-\sqrt{p}}
\]

For \(N = 8\)

\[
f_p(p) = \frac{p^3 \sqrt{p \pi}}{312 \Gamma(3.5)} e^{-\sqrt{p}}
\]

none of which yields a well known pdf and all of which exhibit the same numerical difficulties that we are precisely trying to avoid in this research.
In general (and for $N > 1$) it is very difficult to analytically obtain a closed form for the density (pdf) $f_p(*)$ of $p_N$ of an SIRP $X$. However, we can approximate its distribution (CDF) $F_p(*)$ via Monte Carlo in the following way:

Let the modified $Q_n = (U_n, V_n)$ GOF test, be defined:

$$U_n = \frac{1}{n} \sum_{i} \cos \theta_i |Z_i|$$
$$V_n = \frac{1}{n} \sum_{i} \sin \theta_i |Z_i|$$

with

$$\theta_i = \pi \int_{-\infty}^{m_{i,n}} f_P(t) dt = \pi F_P(m_{i,n})$$

where $m_{i,n}$ is the $i^{th}$ order statistic (David (1990)) from the ordered sample of the corresponding $n$ quadratic forms $p_N$, denoted $p_1 < p_2 < \cdots < p_n$. Let these $n$ samples be obtained from the simulated.

By generating from the same SIRP $X$ used in Phase I we obtain, based on Johnson and Kotz (1970), that the statistic $Q_n = (U_n, V_n)$ fulfills, approximately:

$$\frac{1}{1 - \rho_{uv}^2} \left( \frac{(U_n - E(U_n))^2}{\sigma_u^2} - 2\rho_{uv} \left( \frac{(U_n - E(U_n))(V_n - E(V_n))}{\sigma_u \sigma_v} \right) + \frac{(V_n - E(V_n))^2}{\sigma_v^2} \right) \sim \chi^2_2$$

With this equation we obtain the confidence ellipsoids to implement the empirical GOF tests using the statistic $Q_n^* = (U_n^*, V_n^*)$. 

![Diagram of confidence ellipsoids](image.png)

20. Half

1st. Half
A sample output of the implementation of our approach, showing the Monte Carlo derived $F_{p}^{*}, m_{i,n}^{*}, \delta_{i}^{*}$ for $N = 2$, $n = 10$, $\rho = 0.5$, is presented in Table 1.

<table>
<thead>
<tr>
<th>Obs</th>
<th>$\frac{F_{p}^{<em>}-F_{p}^{ex}}{\sqrt{F_{p}^{</em>}}}$</th>
<th>$m_{i,n}^{*}$</th>
<th>$F_{p}^{<em>}(m_{i,n}^{</em>})$</th>
<th>$\delta_{i}^{*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.683238</td>
<td>0.019328</td>
<td>0.128210</td>
<td>0.402783</td>
</tr>
<tr>
<td>2</td>
<td>0.660789</td>
<td>0.066707</td>
<td>0.224870</td>
<td>0.706450</td>
</tr>
<tr>
<td>3</td>
<td>0.621371</td>
<td>0.151513</td>
<td>0.320800</td>
<td>1.007822</td>
</tr>
<tr>
<td>4</td>
<td>0.560787</td>
<td>0.284144</td>
<td>0.410180</td>
<td>1.288618</td>
</tr>
<tr>
<td>5</td>
<td>0.467949</td>
<td>0.497728</td>
<td>0.501950</td>
<td>1.576921</td>
</tr>
<tr>
<td>6</td>
<td>0.342191</td>
<td>0.833601</td>
<td>0.594480</td>
<td>1.86713</td>
</tr>
<tr>
<td>7</td>
<td>0.245568</td>
<td>1.381543</td>
<td>0.687350</td>
<td>2.159373</td>
</tr>
<tr>
<td>8</td>
<td>0.347593</td>
<td>2.324335</td>
<td>0.778580</td>
<td>2.445980</td>
</tr>
<tr>
<td>9</td>
<td>0.869920</td>
<td>4.265456</td>
<td>0.871280</td>
<td>2.737269</td>
</tr>
<tr>
<td>10</td>
<td>2.337782</td>
<td>10.124611</td>
<td>0.957710</td>
<td>3.008734</td>
</tr>
</tbody>
</table>

Using these values we obtain, through a second Monte Carlo experiment, the empirical estimators of the parameters $E(U_{n}), E(V_{n}), \sigma_{u}^{2}, \sigma_{v}^{2}, \rho_{uv}$, required for implementing the modified $Q_{n}$ GOF test. A sample output of these empirical estimators, corresponding to the values obtained in Table 1, is presented in Table 2. The theoretical value for $Q_{n} = (U_{n}, V_{n})$ is given for comparison.

<table>
<thead>
<tr>
<th>$Q_{n}$</th>
<th>THEORY</th>
<th>MEAN</th>
<th>VARIANCE</th>
<th>CORREL.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_{n}$</td>
<td>-0.200407</td>
<td>-0.199829</td>
<td>0.001842</td>
<td>0.189824</td>
</tr>
<tr>
<td>$V_{n}$</td>
<td>0.363497</td>
<td>0.362216</td>
<td>0.005134</td>
<td>0.189824</td>
</tr>
</tbody>
</table>

Table 3. Results for Exact vs. Modified $Q_{n}$ GOF Tests ($N = 2$)

<table>
<thead>
<tr>
<th>GOF Test</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.05$</th>
<th>$\alpha = 0.01$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>0.0756</td>
<td>0.0386</td>
<td>0.0064</td>
<td>10</td>
</tr>
<tr>
<td>$Q_{n}$</td>
<td>0.0848</td>
<td>0.0398</td>
<td>0.0098</td>
<td>10</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0833</td>
<td>0.0354</td>
<td>0.0068</td>
<td>25</td>
</tr>
<tr>
<td>$Q_{n}$</td>
<td>0.0930</td>
<td>0.0410</td>
<td>0.0092</td>
<td>25</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0936</td>
<td>0.0468</td>
<td>0.0098</td>
<td>50</td>
</tr>
<tr>
<td>$Q_{n}$</td>
<td>0.0912</td>
<td>0.0426</td>
<td>0.0090</td>
<td>50</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0956</td>
<td>0.0450</td>
<td>0.0090</td>
<td>100</td>
</tr>
<tr>
<td>$Q_{n}$</td>
<td>0.0874</td>
<td>0.0444</td>
<td>0.0124</td>
<td>100</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0970</td>
<td>0.0520</td>
<td>0.0080</td>
<td>200</td>
</tr>
<tr>
<td>$Q_{n}$</td>
<td>0.1000</td>
<td>0.0610</td>
<td>0.0170</td>
<td>200</td>
</tr>
</tbody>
</table>
Figure 4: Representation of the Design

Table 7. Descriptive Statistics for The $Q_n$ Data in Table 6.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.05$</th>
<th>$\alpha = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0884</td>
<td>0.0439</td>
<td>0.0104</td>
</tr>
<tr>
<td>Median</td>
<td>0.0880</td>
<td>0.0435</td>
<td>0.0106</td>
</tr>
<tr>
<td>Std-Dev.</td>
<td>0.0054</td>
<td>0.0034</td>
<td>0.0020</td>
</tr>
<tr>
<td>Min</td>
<td>0.0808</td>
<td>0.0398</td>
<td>0.0072</td>
</tr>
<tr>
<td>Max</td>
<td>0.1002</td>
<td>0.0610</td>
<td>0.0170</td>
</tr>
<tr>
<td>$Q_1$</td>
<td>0.0841</td>
<td>0.0422</td>
<td>0.0090</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.0923</td>
<td>0.0450</td>
<td>0.0116</td>
</tr>
<tr>
<td>$L_B$</td>
<td>0.0866</td>
<td>0.0428</td>
<td>0.0095</td>
</tr>
<tr>
<td>$U_B$</td>
<td>0.0902</td>
<td>0.0450</td>
<td>0.0111</td>
</tr>
</tbody>
</table>
Table 4. Modified \(Q_n\) GOF Test Results For \(N=8\).

<table>
<thead>
<tr>
<th>GOF Test</th>
<th>(\alpha = 0.1)</th>
<th>(\alpha = 0.05)</th>
<th>(\alpha = 0.01)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q_n)</td>
<td>0.08980</td>
<td>0.04200</td>
<td>0.00720</td>
<td>25</td>
</tr>
<tr>
<td>(Q_n)</td>
<td>0.08648</td>
<td>0.04400</td>
<td>0.00972</td>
<td>50</td>
</tr>
<tr>
<td>(Q_n)</td>
<td>0.08476</td>
<td>0.04504</td>
<td>0.01160</td>
<td>100</td>
</tr>
<tr>
<td>(Q_n)</td>
<td>0.08220</td>
<td>0.04160</td>
<td>0.01190</td>
<td>200</td>
</tr>
</tbody>
</table>

We investigated further how close these two GOF tests (Exact \(E\) and modified \(Q_n\)) really are. We took their percent rejection difference. That is, for every pair \((E, Q_n)\) with the same setting \((n, 2, \rho)\), we obtained the difference percent rejections:

\[
\delta_1 = P_r(E) - P_r(Q_n)
\]

The descriptive statistics corresponding to these \(\delta_1\) values are presented in Table 5. Notice that this analysis is performed for \(N = 2\) only and includes correlations \(\rho = 0.0, 0.5, 0.9\).

Table 5. Descriptive Statistics for \(\delta_1 = P_r(E) - P_r(Q_n)\).

<table>
<thead>
<tr>
<th>Statistic</th>
<th>(\alpha = 0.1)</th>
<th>(\alpha = 0.05)</th>
<th>(\alpha = 0.01)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0039</td>
<td>0.0013</td>
<td>-0.0023</td>
</tr>
<tr>
<td>Median</td>
<td>0.0098</td>
<td>0.0020</td>
<td>-0.0020</td>
</tr>
<tr>
<td>Std-Dev.</td>
<td>0.0132</td>
<td>0.0057</td>
<td>0.0013</td>
</tr>
<tr>
<td>Min</td>
<td>-0.0162</td>
<td>-0.0096</td>
<td>-0.0040</td>
</tr>
<tr>
<td>Max</td>
<td>0.0171</td>
<td>0.0080</td>
<td>0.0002</td>
</tr>
<tr>
<td>(Q_1)</td>
<td>-0.0100</td>
<td>-0.0025</td>
<td>-0.0035</td>
</tr>
<tr>
<td>(Q_3)</td>
<td>0.0161</td>
<td>0.0069</td>
<td>-0.0014</td>
</tr>
<tr>
<td>(L_B)</td>
<td>-0.0045</td>
<td>-0.0023</td>
<td>-0.0031</td>
</tr>
<tr>
<td>(U_B)</td>
<td>0.0123</td>
<td>0.0049</td>
<td>-0.0015</td>
</tr>
</tbody>
</table>
Table 6. Results for the General Factorial Experimental Design.

<table>
<thead>
<tr>
<th>Test</th>
<th>$n$</th>
<th>$N$</th>
<th>$\rho$</th>
<th>$\alpha = 0.1$</th>
<th>$\alpha = 0.05$</th>
<th>$\alpha = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>25</td>
<td>2</td>
<td>0.0</td>
<td>0.0826</td>
<td>0.0402</td>
<td>0.0064</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>25</td>
<td>2</td>
<td>0.0</td>
<td>0.0930</td>
<td>0.0430</td>
<td>0.0094</td>
</tr>
<tr>
<td>Exact</td>
<td>25</td>
<td>2</td>
<td>0.5</td>
<td>0.0774</td>
<td>0.0352</td>
<td>0.0050</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>25</td>
<td>2</td>
<td>0.5</td>
<td>0.0936</td>
<td>0.0448</td>
<td>0.0090</td>
</tr>
<tr>
<td>Exact</td>
<td>25</td>
<td>2</td>
<td>0.9</td>
<td>0.0870</td>
<td>0.0414</td>
<td>0.0062</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>25</td>
<td>2</td>
<td>0.9</td>
<td>0.0936</td>
<td>0.0429</td>
<td>0.0102</td>
</tr>
<tr>
<td>Exact</td>
<td>50</td>
<td>2</td>
<td>0.5</td>
<td>0.0949</td>
<td>0.0465</td>
<td>0.0094</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>50</td>
<td>2</td>
<td>0.5</td>
<td>0.0880</td>
<td>0.0450</td>
<td>0.0092</td>
</tr>
<tr>
<td>Exact</td>
<td>100</td>
<td>2</td>
<td>0.0</td>
<td>0.0978</td>
<td>0.0466</td>
<td>0.0098</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>100</td>
<td>2</td>
<td>0.0</td>
<td>0.0850</td>
<td>0.0440</td>
<td>0.0116</td>
</tr>
<tr>
<td>Exact</td>
<td>100</td>
<td>2</td>
<td>0.5</td>
<td>0.0963</td>
<td>0.0489</td>
<td>0.0096</td>
</tr>
<tr>
<td>$Q_n$</td>
<td>100</td>
<td>2</td>
<td>0.5</td>
<td>0.0824</td>
<td>0.0409</td>
<td>0.0109</td>
</tr>
<tr>
<td>Exact</td>
<td>100</td>
<td>2</td>
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Table 8. Effect of Covariance Matrix Estimation in $P_r$.  

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Table 9. Descriptive Statistics for $\delta_2 = P_r(\Sigma) - P_r(\Sigma^*)$.  

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Estimated/exact results are similar when adequate minimum sample sizes:

- \( N = 2, 4, \) and \( 8 \)
- Provided for \( N \) = \( f(N) \)
- Minimum size \( n = f(N) \)
- Significant effect of vector size \( (N) \)
- Significant effect of sample size \( (n) \)
- Small effect of inter-variate correlation

Experimental Results
Conclusions