Further Monte Carlo Investigations of a Model for Non Gaussian Radar Clutter Generation

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Outline

- Research Problem Statement
- Problem Background
- The Bivariate Case
- Parameter Estimation and Validation
- General Multivariate Case
- Experimentation Results
- Conclusions

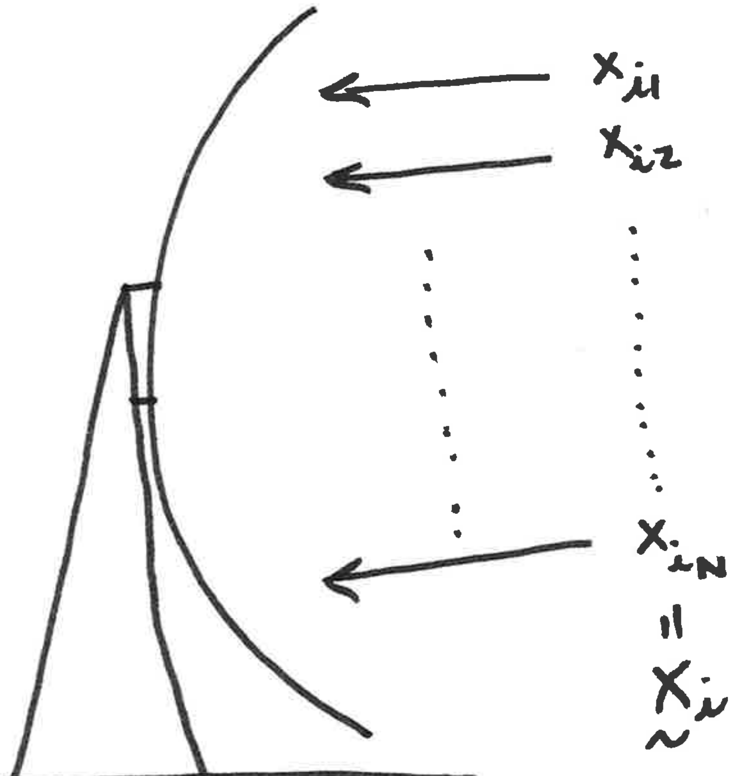
Research Problem:

output:

Distribution Identification through $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$

input:

$\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots \underset{\sim}{X}_n$



Process: synthesige $\hat{\Sigma}$ and $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$

where:

$$\hat{p}_i = \underset{\sim}{x}_i' \, \hat{\Sigma}^{-1} \underset{\sim}{x}_i$$

$$1 \leq i \leq n$$

Spherically Invariant Random Processes:

$$X = S * Z$$

where: $Z \sim MVN_N(0, M)$

S is a univariate (indep.) process

let $\mathbb{E}(S^2) = 1$ (by so defining the S)

Then $\Sigma_X = \Sigma = M$ (Variance of X)

let the quadratic form $p = X' \Sigma^{-1} X$

then, the conditional pdf of $X|S$ is:

$$f_{X|S}(x|s) = (2\pi)^{-\frac{N}{2}} |M|^{-\frac{1}{2}} s^{-N} \exp\left(\frac{-p}{2s^2}\right)$$

and $f(x) = (2\pi)^{-\frac{N}{2}} |M|^{-\frac{1}{2}} h_N(p)$

$$\text{where: } h_N(p) = \int_0^{+\infty} s^{-N} \exp\left(\frac{-p}{2s^2}\right) f_S(s)\, ds$$

with $f_S(s)$ the pdf of the process S

Finally, the quadratic form p will have:

$$f_P(p) = \frac{1}{2^{\frac{N}{2}} \Gamma(\frac{N}{2})} p^{\frac{N}{2}-1} h_N(p)$$

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In particular, the K-Distributed SIRP X is:

$$f_X(x) = \frac{2b}{\Gamma(\alpha)} \left(\frac{bx}{2}\right)^{\alpha} K_{\alpha-1}(bx)$$

where α, b are shape and scale parameters and K_N is the N^{th} order Modified Bessel Funct. (2° kind)

For such K-Distributed SIRP X we have:

$$f_S(s) = \frac{2}{\Gamma(\alpha) 2^{\alpha}} (bs)^{2\alpha-1} \exp\left\{\frac{-b^2 s^2}{2}\right\}$$

$$h_N(p) = \frac{b^N}{\Gamma(\alpha)} \frac{(b\sqrt{p})^{\alpha-\frac{N}{2}}}{2^{\alpha-1}} K_{\frac{N}{2}-\alpha}(b\sqrt{p}) \quad *$$

* for long tailed K-Dist X, α is very small.

For univariate Laplace SIRP X:

$$f_X(x) = \frac{1}{2\lambda} \exp\left\{-\frac{|x-\mu|}{\lambda}\right\}; \qquad \lambda > 0$$

To generate a suitable X via Monte Carlo:

$w \sim \exp(1)$

$y = \sqrt{2w} \sim$ Rayleigh (with $\mathbb{E}(y) = 2$)

$s = \frac{y}{\sqrt{2}}$ is suitable for M.C.

i) $\mathbb{E}(s) = 1$ and ii) $f_S(s) = 2s \exp(-s^2)$

then: $x = s \cdot z$ (with $z \sim N(0,1)$)

$p = x'x = x^2$

$h_N(p) = \sqrt{\pi} \exp(-\sqrt{2p})$

$f_p(p) = \frac{1}{\sqrt{2p}} \exp(-\sqrt{2p})$

then, Transforming: $t = \sqrt{2p} \sim \exp(1)$ we find:

we can test the SIRP X:

* $f_X(x) = \sqrt{2\pi} \, |\Sigma|^{-\frac{1}{2}} h_N(p) = \frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|)$

Laplace w/ $\lambda = \frac{1}{\sqrt{2}}$

* the quadratic function p (indirectly via t)

this is precisely the problem of M.C. experiments:

* for higher dimensions
* for $\alpha \neq 1$ (Shape parameter of K-Distr.)

* we are currently investigating for $0 < \alpha < 1$
* and for estimators of Σ.
* $p = x'x$ Serves for distribution identification

Our analysis yielded the conveniently tractable special case of the K-Distributed bivariate ($N = 2$) SIRP $X = s * Z$, with shape parameter $\alpha = N/2 - 0.5$ and scale parameter $b = 1$. This special SIRP has a quadratic form p with closed form pdf function $f_P(*)$:

$$Given \quad h_N(p) = \frac{\sqrt{p}^{N/2-0.5-N/2}}{\Gamma(N/2-0.5)2^{N/2-0.5-1}} \times K_{N/2-N/2+0.5}(\sqrt{p}) = \sqrt{\frac{\pi}{p}}\frac{exp(-\sqrt{p})}{\Gamma(\frac{N-1}{2})2^{\frac{N-2}{2}}}$$

$$and \quad K_{\frac{N}{2}-\alpha} = K_{0.5}(x) = K_{-0.5}(x) = \sqrt{\frac{\pi}{2}} \times \sqrt{\frac{1}{x}}exp(-x)$$

$$Then \quad f_P(p) = \frac{1}{2^1\Gamma(1)} \times p^{1-1}h_2(p) = \frac{1}{2}p^{-1/2}exp(-\sqrt{p})$$

These functions will provide the theoretical comparison values to validate our Monte Carlo study of the SIRP process $X = s * Z$. However, in the above form, this pdf is still too complex for us to work with, directly. And certain transformations are required to simplify our work.

Under the transformation $\frac{w^2}{2} = \sqrt{p}$, the resulting random variable (r.v.) w is distributed Rayleigh and easy to test for GOF. Such GOF test is necessary (i) to validate our Monte Carlo experiment and (ii) to compare the Power of our modified Q_n test with an established GOF test for p. It is statistically equivalent to test GOF directly on the r.v. p or on its transformation, the r.v. w. For, if $p \sim F_P$ then $w \sim$ Rayleigh. We select the second alternative for its ease and computational speed.

Accordingly, the pdf $f_S(*)$ of the driver random variable $s > 0$, for this special case of K-Distributed SIRP $X = s * Z$ is:

$$f_S(s) = \frac{2}{\Gamma(\alpha)2^\alpha}s^{2\alpha-1}exp(\frac{-s^2}{2}) = \sqrt{\frac{2}{\pi}}exp(-s^2/2)$$

which, under the transformation $y = s^2/2$ becomes:

$$f_Y(y) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{y}}exp(-y) = \frac{1}{\Gamma(1/2)}y^{\frac{-1}{2}}exp(-y)$$

i.e., a Gamma distribution with parameters $\lambda = 1, r = 1/2$. It is now easier and faster to generate a Gamma r.v. y and obtain $s = \sqrt{2y}$.

The above random variable s also has the convenient property that:

$$E(s^2) = \int_0^\infty s^2\sqrt{\frac{2}{\pi}}exp(-s^2/2)\,ds = 1$$

therefore, insuring that the resulting covariances ($\Sigma = M$) of our Gaussian (Z) and SIRP $X = s * Z$ processes are equal.

We also investigated other special cases of SIRP for $\alpha = N/2 - 0.5$ and $b = 1$:

$$For \quad N = 3 \qquad f_P(p) = \frac{\sqrt{\pi}}{4\Gamma(1.5)}e^{-\sqrt{p}}$$

$$For \quad N = 4 \qquad f_P(p) = \frac{\sqrt{p\pi}}{8\Gamma(2)}e^{-\sqrt{p}}$$

$$For \quad N = 8 \qquad f_P(p) = \frac{p^2\sqrt{p\pi}}{3!2^7\Gamma(3.5)}e^{-\sqrt{p}}$$

none of which yields a well known pdf and all of which exhibit the same numerical difficulties that we are precisely trying to avoid in this research.

In general (and for $N > 1$) it is very difficult to analytically obtain a closed form for the density (pdf) $f_P(*)$ of p_N of an SIRP X. However, we can approximate its distribution (CDF) $F_P(*)$ via Monte Carlo in the following way:

Let the **modified** $Q_n = (U_n, V_n)$ GOF test, be defined:

$$U_n = \frac{1}{n}\sum_{i}^{n} cos\theta_i \, |Z_i|$$

$$V_n = \frac{1}{n}\sum_{i}^{n} sin\theta_i \, |Z_i|$$

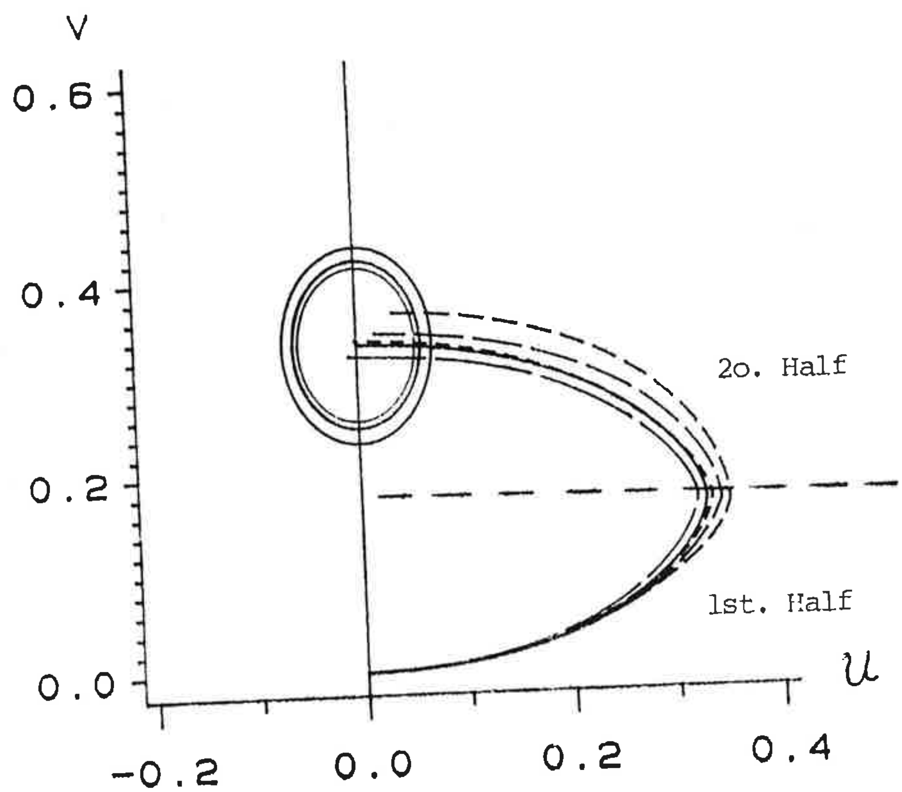
$$\text{with} \qquad \theta_i = \pi \int_{-\infty}^{m_{i:n}} f_P(t)dt = \pi F_P(m_{i:n})$$

where $m_{i:n}$ is the i^{th} order statistic (David (1990)) from the ordered sample of the corresponding n quadratic forms p_N, denoted $p_1 < p_2 < \cdots < p_n$. Let these n samples be obtained from the simulated,

By generating from the same SIRP X used in Phase I we obtain, based on Johnson and Kotz (1970), that the **statistic $Q_n = (U_n, V_n)$ fulfills, approximately:**

$$\frac{1}{1-\rho_{uv}^2}\{\frac{(U_n - E(U_n))^2}{\sigma_u^2} - 2\rho_{uv}\frac{(U_n - E(U_n))(V_n - E(V_n))}{\sigma_u \sigma_v} + \frac{(V_n - E(V_n))^2}{\sigma_v^2}\} \quad \sim \quad \chi_2^2$$

With this equation we obtain the confidence ellipsoids to implement the empirical GOF tests using the statistic $Q_n^* = (U_n^*, V_n^*)$.



A sample output of the implementation of our approach, showing the Monte Carlo derived $F_P^*, m_{i:n}^*, \theta_i^*$ for $N = 2$, $n = 10$, $\rho = 0.5$, is presented in Table 1.

Table 1. Sample of Empirical Values for $N = 2$, $\rho = 0.5$:

i^{th} *Obs*	$\lvert\frac{p - p_{avg}}{s}\rvert$	$m_{i:n}^*$	$F_P^*(m_{i:m}^*)$	θ_i^*
1	0.683238	0.019328	0.128210	0.402783
2	0.660789	0.066707	0.224870	0.706450
3	0.621371	0.151513	0.320800	1.007822
4	0.560787	0.284144	0.410180	1.288618
5	0.467949	0.497728	0.501950	1.576921
6	0.342191	0.833601	0.594480	1.867613
7	0.245568	1.381543	0.687350	2.159373
8	0.347593	2.324335	0.778580	2.445980
9	0.869920	4.265456	0.871280	2.737206
10	2.337782	10.124611	0.957710	3.008734

Using these values we obtain, through a second Monte Carlo experiment, the empirical estimators of the parameters $E(U_n), E(V_n), \sigma_u^2, \sigma_v^2, \rho_{uv}$, required for implementing the modified Q_n GOF test. A sample output of these empirical estimators, corresponding to the values obtained in Table 1, is presented in Table 2. The theoretical value for $Q_n = (U_n, V_n)$ is given for comparison.

Table 2. Corresponding Simulation Results for $N = 2$, $\rho = 0.5$:

Q_n	THEORY	MEAN	VARIANCE	CORREL.
U_n	-0.200407	-0.199829	0.001842	0.189824
V_n	0.363497	0.362216	0.005134	0.189824

Table 3. Results for Exact vs. Modified Q_n GOF Tests ($N = 2$)

GOF Test	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	n
Exact	0.0756	0.0386	0.0064	10
Q_n	0.0848	0.0398	0.0098	10
Exact	0.0838	0.0354	0.0068	25
Q_n	0.0930	0.0410	0.0092	25
Exact	0.0936	0.0468	0.0098	50
Q_n	0.0912	0.0426	0.0090	50
Exact	0.0956	0.0450	0.0090	100
Q_n	0.0874	0.0444	0.0124	100
Exact	0.0970	0.0520	0.0080	200
Q_n	0.1000	0.0610	0.0170	200

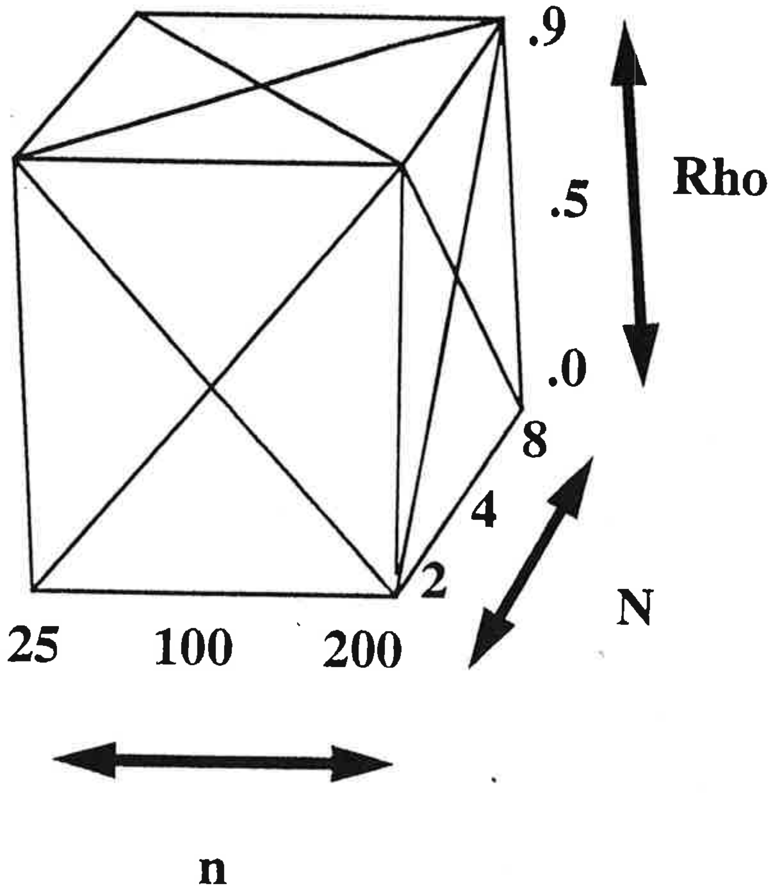


Figure 4: Representation of the Design

Table 7. Descriptive Statistics for The Q_n Data in Table 6.

Statistic	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Mean	0.0884	0.0439	0.0104
Median	0.0880	0.0435	0.0106
Std-Dev.	0.0054	0.0034	0.0020
Min	0.0808	0.0398	0.0072
Max	0.1002	0.0610	0.0170
Q_1	0.0841	0.0422	0.0090
Q_3	0.0923	0.0450	0.0116
L_B	0.0866	0.0428	0.0095
U_B	0.0902	0.0450	0.0111

Table 4. Modified Q_n GOF Test Results For N=8.

GOF Test	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$	n
Q_n	0.08980	0.04200	0.00720	25
Q_n	0.08648	0.04400	0.00972	50
Q_n	0.08476	0.04504	0.01160	100
Q_n	0.08220	0.04160	0.01190	200

We investigated further how close these two GOF tests (Exact (E) and modified Q_n) really are. We took their percent rejection difference. That is, for every pair (E, Q_n) with the same setting $(n, 2, \rho)$, we obtained the difference percent rejections:

$$\delta_1 = P_r(E) - P_r(Q_n)$$

The descriptive statistics corresponding to these δ_1 values are presented in Table 5. Notice that this analysis is performed for $N = 2$ only and includes correlations $\rho = 0.0, 0.5, 0.9$.

Table 5. Descriptive Statistics for $\delta_1 = P_r(E) - P_r(Q_n)$.

Statistic	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Mean	0.0039	0.0013	-0.0023
Median	0.0098	0.0020	-0.0020
Std-Dev.	0.0132	0.0057	0.0013
Min	-0.0162	-0.0096	-0.0040
Max	0.0171	0.0080	0.0002
Q_1	-0.0100	-0.0025	-0.0035
Q_3	0.0161	0.0069	-0.0014
L_B	-0.0045	-0.0023	-0.0031
U_B	0.0123	0.0049	-0.0015

Table 6. Results for the General Factorial Experimental Design.

Test	n	N	ρ	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Exact	25	2	0.0	0.0826	0.0402	0.0064
Q_n	25	2	0.0	0.0930	0.0430	0.0094
Exact	25	2	0.5	0.0774	0.0352	0.0050
Q_n	25	2	0.5	0.0936	0.0448	0.0090
Exact	25	2	0.9	0.0870	0.0414	0.0062
Q_n	25	2	0.9	0.0936	0.0429	0.0102
Exact	50	2	0.5	0.0949	0.0465	0.0094
Q_n	50	2	0.5	0.0880	0.0450	0.0092
Exact	100	2	0.0	0.0978	0.0466	0.0098
Q_n	100	2	0.0	0.0850	0.0440	0.0116
Exact	100	2	0.5	0.0983	0.0489	0.0096
Q_n	100	2	0.5	0.0824	0.0409	0.0109
Exact	100	2	0.9	0.0968	0.0494	0.0095
Q_n	100	2	0.9	0.0828	0.0416	0.0115
Exact	200	2	0.0	0.0984	0.0460	0.0090
Q_n	200	2	0.0	0.0839	0.0442	0.0123
Exact	200	2	0.5	0.0970	0.0482	0.0096
Q_n	200	2	0.5	0.0808	0.0412	0.0109
Exact	200	2	0.9	0.1006	0.0508	0.0103
Q_n	200	2	0.9	0.0813	0.0423	0.0125
Q_n	25	4	0.0	0.0918	0.0426	0.0075
Q_n	25	4	0.5	0.1002	0.0453	0.0081
Q_n	25	4	0.9	0.0894	0.0429	0.0079
Q_n	50	4	0.5	0.0942	0.0477	0.0106
Q_n	100	4	0.0	0.0917	0.0465	0.0129
Q_n	100	4	0.5	0.0872	0.0454	0.0113
Q_n	100	4	0.9	0.0879	0.0453	0.0109
Q_n	200	4	0.5	0.0832	0.0432	0.0133
Q_n	25	8	0.0	0.0893	0.0430	0.0078
Q_n	25	8	0.5	0.0928	0.0439	0.0074
Q_n	25	8	0.9	0.0898	0.0438	0.0085
Q_n	50	8	0.5	0.0842	0.0426	0.0096
Q_n	100	8	0.0	0.0808	0.0408	0.0106
Q_n	100	8	0.5	0.0890	0.0442	0.0112
Q_n	100	8	0.9	0.0854	0.0434	0.0113
Q_n	200	8	0.5	0.0822	0.0416	0.0119

Table 8. Effect of Covariance Matrix Estimation in P_r.

Σ	n	N	ρ	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Known	10	2	0.5	0.08480	0.03980	0.00980
Estim	10	2	0.5	0.14420	0.07840	0.02060
Known	25	2	0.5	0.09300	0.04100	0.00920
Estim	25	2	0.5	0.10020	0.05780	0.01500
Known	50	2	0.5	0.09120	0.04260	0.00900
Estim	50	2	0.5	0.08300	0.04120	0.01180
Known	100	2	0.5	0.08740	0.04440	0.01240
Estim	100	2	0.5	0.07660	0.03620	0.00820
Known	200	2	0.5	0.10000	0.06100	0.01700
Estim	200	2	0.5	0.09000	0.04100	0.01200
Known	25	4	0.5	0.10020	0.04530	0.00810
Estim	25	4	0.5	0.22920	0.14430	0.05130
Known	100	4	0.5	0.08720	0.04540	0.01136
Estim	100	4	0.5	0.10380	0.05876	0.01724
Known	25	8	0.5	0.08980	0.04200	0.00720
Estim	25	8	0.5	0.94400	0.86200	0.55580
Known	50	8	0.5	0.08648	0.04400	0.00972
Estim	50	8	0.5	0.52872	0.38176	0.17040
Known	100	8	0.5	0.08476	0.04504	0.01160
Estim	100	8	0.5	0.28084	0.18348	0.06648
Known	200	8	0.5	0.08220	0.04160	0.01190
Estim	200	8	0.5	0.16090	0.09290	0.02830

Table 9. Descriptive Statistics for $\delta_2 = P_r(\Sigma) - P_r(\Sigma^*)$.

Statistic	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
Mean	-0.1595	-0.1351	-0.0763
Median	-0.0594	-0.0386	-0.0108
Std-Dev	0.2666	0.2490	0.1637
Max	0.0108	0.0200	0.0050
Min	-0.8542	-0.8200	-0.5486
Q_1	-0.1961	-0.1384	-0.0549
Q_3	0.0082	0.0014	-0.0028
L_B	-0.3387	-0.3024	-0.1864
U_B	0.0197	0.0322	0.0337

Experimental Results

- small effect of inter-variate correlation
- significant effect of sample size (n)
- significant effect of vector size (N)
- minimum size $n = f(N)$
- provided for $N = 2, 4$ and 8
- when adequate minimum sample sizes: estimated/exact results are similar

Conclusions

- General Procedure works well
- useful in distribution identification
- useful in evaluation of alternative estimators of process covariance matrix Sigma
- useful in providing minimum sample sizes
- proposed future work: parameter bank for incoming signal distribution identification