A CONFIGURATION OF THE TRANSPOSITIONS IN $S_{2n}$

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Abstract. We solve a problem concerning a combinatorial design involving the transpositions in the symmetric group $S_{2n}$. The problem admits equivalent formulations in terms of symmetric permutation matrices and 1-factorizations of complete graphs.

1. Introduction and Preliminaries

The symmetric group $S_{2n}$ on $2n$ symbols contains exactly

$$\binom{2n}{2} = \frac{2n(2n-1)}{2} = n(2n-1)$$

transpositions. For example, $S_4$ contains the six transpositions

$(12), (13), (14), (23), (24), \text{ and } (34)$.

Problem 1(a): Find elements $\theta_1, \ldots, \theta_{2n-1} \in S_{2n}$ satisfying the following two statements.

(1) $\theta_i = (a_{i1}b_{i1})\cdots(a_{in}b_{in}), 1 \leq i \leq 2n-1$, is a product of $n$ disjoint transpositions. (Thus each of the numbers in $\{1, 2, \ldots, 2n\}$ occurs exactly once in such an expression.)

(2) Each transposition in $S_{2n}$ occurs exactly once in the expressions giving $\{\theta_1, \ldots, \theta_{2n-1}\}$.

Example 1.1. If $n = 2$ and $n = 3$, then it is easy to solve Problem 1(a). For example, when $n = 2$ and $n = 3$, respectively, we can take

$\theta_1 = (12)(34)$

$\theta_2 = (13)(24)$

$\theta_3 = (14)(23)$

and

$\theta_1 = (12)(34)(56)$

$\theta_2 = (13)(25)(46)$

$\theta_3 = (14)(26)(35)$

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Before pursuing this problem further, we now find two alternative ways to formulate this problem.

To each permutation \( \sigma \in S_m \), we can associate an \( m \times m \) matrix \( P_\sigma \) whose \((i,j)\)-entry is defined by

\[
P_\sigma(ij) = \begin{cases} 
1 & \text{if } \sigma(i) = j \\
0 & \text{if } \sigma(i) \neq j.
\end{cases}
\]

The matrix \( P_\sigma \) has exactly one 1 in each row and column, while the remaining entries equal zero. Such a matrix is called a permutation matrix. We state the following lemma without proof, as it is easy to verify.

**Lemma 1.2.**

1. A permutation matrix \( P_\sigma \) is a symmetric matrix if and only if \( \sigma \) is a product of disjoint transpositions.
2. A \( 2n \times 2n \) permutation matrix \( P_\sigma \) is a symmetric matrix with zeros in each entry of the main diagonal if and only if \( \sigma \) is a product of \( n \) disjoint transpositions.

Let \( I_m \) denote the \( m \times m \) identity matrix and let \( J_m \) denote the \( m \times m \) matrix all of whose entries equal 1.

**Problem 1(b):** Find symmetric permutation matrices \( P_{\sigma_i}, 1 \leq i \leq 2n-1 \), such that \( J_{2n} = I_{2n} + \sum_{i=1}^{2n-1} P_{\sigma_i} \).

For such an equation to hold, each diagonal entry of \( P_{\sigma_i} \) must equal zero. Thus each \( \sigma_i \) would be a product of \( n \) disjoint transpositions by Lemma 1.2 (2). It follows that the \( n(2n-1) \) transpositions occurring in \( P_{\sigma_1}, \ldots, P_{\sigma_{2n-1}} \) would be precisely the \( n(2n-1) \) distinct transpositions in \( S_{2n} \). Therefore Problems 1(a) and 1(b) are equivalent.

Our third version of these problems involves finding 1-factorizations of complete graphs. A complete graph \( K_m \) is a graph with \( m \) vertices such that every pair of vertices is adjacent. In other words, each pair of vertices determines an edge and thus there are exactly \( \binom{m}{2} \) edges in \( K_m \).

Suppose that a graph \( G \) has \( 2n \) vertices. A 1-factor of \( G \) is a subgraph of \( G \) having the same vertices as \( G \) and containing exactly \( n \) disjoint edges of \( G \) such that each of the vertices lies on exactly one of these \( n \) edges. A 1-factorization of \( G \) is a partition of \( G \) into disjoint 1-factors of \( G \). See [H, Chapter 9] for more details of these ideas.
Problem 1(c): Find a 1-factorization of the complete graph $K_{2n}$.

We label the vertices of $K_{2n}$ as $1, 2, 3, \ldots, 2n$. We identify the edge \{i, j\} with the transposition $(ij)$ in the symmetric group $S_{2n}$. A 1-factor of $K_{2n}$ becomes a product of $n$ disjoint transpositions. We now see that finding a 1-factorization of $K_{2n}$ is equivalent to solving Problem 1(a). Thus Problems 1(a) and 1(c) are equivalent.

Example 1.3. The figure below shows $K_6$ and its 1-factorization corresponding to the permutations $\theta_1, \ldots, \theta_5$ from Example 1.1.

![Graph example](image)

We will show that all three (equivalent) problems have solutions. Moreover we will give two proofs of this result. We begin with a constructive solution to Problem 1(a).

2. First solution of Problem 1(a)

Let $\Sigma(A)$ denote the group of permutations of a set $A$. Thus $S_n \cong \Sigma(\{1, 2, \ldots, n\})$.

The first solution of Problem 1(a) is by induction on $2n$, $n \geq 1$. If $2n = 2$, the transposition $(1 \ 2)$ satisfies all the desired conditions. Now assume that $2n > 2$.

**Case 1: $n$ is even.** Since $1 \leq n/2 < n$ and $2(n/2) - 1 = n - 1$, the induction assumption applied to $2(n/2)$ implies that there are

$$\theta_1', \ldots, \theta_{n-1}' \in \Sigma(\{1, 2, \ldots, n\})$$

and

$$\theta_1'', \ldots, \theta_{n-1}'' \in \Sigma(\{n + 1, n + 2, \ldots, 2n\})$$
satisfying properties (1) and (2). Let \( \theta_i = \theta'_i \theta''_i \) for \( 1 \leq i \leq n - 1 \). Next, for \( 0 \leq k \leq n - 1 \) and \( 1 \leq i \leq n \), let

\[
\sigma_{k,i} = \begin{cases} 
  i + n + k & \text{if } 1 \leq i + k \leq n, \\
  i + k & \text{if } n + 1 \leq i + k \leq 2n - 1.
\end{cases}
\]

Thus \( n + 1 \leq \sigma_{k,i} \leq 2n \) for all \( i \) and \( k \). Let

\[
\theta_{n+k} = (1 \sigma_{k,1})(2 \sigma_{k,2}) \ldots (n \sigma_{k,n}) \text{ for } 0 \leq k \leq n - 1.
\]

The transpositions in \( \theta_{n+k} \) are disjoint because the set of first entries equals \( \{1, \ldots, n\} \) and the set of second entries equals \( \{n + 1, \ldots, 2n\} \). Furthermore, the transpositions occurring in \( \theta_{n+i} \) and \( \theta_{n+k}, j \neq k \), are disjoint because \( \sigma_{k,i} - i \equiv k \mod n \). Thus, the \( \theta_i \) satisfy properties (1) and (2) of Problem 1(a). See Example 2.1 below for an illustration of this construction.

**Case 2: \( n \) is odd.** Since \( (n+1) < 2n \) and \( 2((n+1)/2) - 1 = n \), the inductive assumption applied to \( 2(n+1)/2 \) implies that there are

\[
\theta'_1, \ldots, \theta'_n \in \Sigma(\{1, 2, \ldots, n, *\})
\]

and

\[
\theta''_1, \ldots, \theta''_n \in \Sigma(\{n + 1, n + 2, \ldots, 2n, *\})
\]

satisfying properties (1) and (2). Relabeling if necessary, we may assume that \( \theta'_i \) contains the transposition \((i, *)\) and \( \theta''_i \) contains the transposition \((n + i, *)\) for each \( i = 1, \ldots, n \). We obtain \( \theta_i \) by concatenating \( \theta'_i \) and \( \theta''_i \) and replacing \((i, *)(n + i, *)\) with the transposition \((i, n + i)\).

For the remaining transpositions, use the same \( \theta_{n+k} \) as in Case 1, but this time for \( 1 \leq k \leq n - 1 \). Using the same reasoning as in Case 1, we see that the \( \theta_i \) satisfy properties (1) and (2) of Problem 1(a). See again Example 2.1 below.

**Example 2.1.** If \( n = 2 \), we have

\[ \theta'_1 = (12) \text{ and } \theta''_1 = (34), \text{ so } \theta_1 = (12)(34). \]

Furthermore,

\[ \sigma_{0,1} = 3, \ \sigma_{0,2} = 4, \ \sigma_{1,1} = 4 \text{ and } \sigma_{1,2} = 3, \]

so

\[ \theta_2 = (13)(24) \text{ and } \theta_3 = (14)(23). \]

If \( n = 3 \), we have

\[ \theta'_1 = (1*)(23), \ \theta''_1 = (4*)(56). \]

\[ \theta'_2 = (13)(2*), \ \theta''_2 = (46)(5*), \]

\[ \theta'_3 = (12)(3*), \ \theta''_3 = (45)(6*), \]
Thus,

\[ \theta_1 = (14)(23)(56), \theta_2 = (13)(25)(46) \text{ and } \theta_3 = (12)(36)(45). \]

The remaining two are constructed as in Case 1, and so

\[ \theta_4 = (15)(26)(34) \text{ and } \theta_5 = (16)(24)(35). \]

3. Second solution of Problem 1(a)

We now use group theory to solve Problem 1(a). See any introductory text on group theory such as [R] for basic information on conjugacy classes, cycle decompositions of permutations, and groups acting on sets.

**Lemma 3.1.** Let \( \sigma \in S_{2n} \) be a \((2n - 1)\)-cycle and let \( \tau \in S_{2n} \) be a transposition. Let \( G = \langle \sigma \rangle \) be the cyclic group generated by \( \sigma \) and let \( G \) act on \( S_{2n} \) by conjugation. Then the orbit of \( \tau \) has cardinality \( 2n - 1 \).

**Proof.** We will show that the elements \( \{\sigma^i\tau\sigma^{-i} \mid 0 \leq i \leq 2n - 2\} \) are distinct. Suppose that \( \sigma^i\tau\sigma^{-i} = \sigma^j\tau\sigma^{-j} \) where \( 0 \leq j \leq i \leq 2n - 2 \). Then \( \sigma^k\tau\sigma^{-k} = \tau \) where \( k = i - j \) and \( 0 \leq k \leq 2n - 2 \). We know that \( \sigma^k \) has order \((2n - 1)/\gcd(2n - 1, k)\). Letting \( d = \gcd(2n - 1, k) \), it follows that \( \sigma^k \) is a product of \( d \) disjoint \( e \)-cycles where \( de = 2n - 1 \). Writing \( \tau = (rs) \) and using the fact that \( \sigma^k\tau\sigma^{-k} = \tau \), it follows that either \( \sigma^k(r) = r \) and \( \sigma^k(s) = s \), in which case \( e = 1 \), or \( \sigma^k(r) = s \) and \( \sigma^k(s) = r \), in which case \( e = 2 \). As \( de = 2n - 1 \), we have \( e = 1 \) and \( d = 2n - 1 \). It follows that \( \sigma^k = 1 \) and thus \( k = 0 \) and \( i = j \). \( \square \)

**Lemma 3.2.** Let \( \sigma = (1\ 2\ 3\ \cdots\ 2n - 1) \in S_{2n} \) be a \((2n - 1)\)-cycle, and let \( G = \langle \sigma \rangle \) be the cyclic group generated by \( \sigma \). Let \( \tau_j = (j, 2n - j + 1), 1 \leq j \leq n \). Thus \( \tau_1, \ldots, \tau_n \) are the transpositions

\[ (1, 2n), (2, 2n - 1), (3, 2n - 2), \ldots, (n, n + 1). \]

Let \( G \) act on \( S_{2n} \) by conjugation. Then \( \tau_1, \ldots, \tau_n \) are in disjoint orbits.

**Proof.** For \( 1 \leq j \leq n \) and \( 0 \leq i \leq 2n - 2 \), we have

\[ \sigma^i\tau_j\sigma^{-i} = (\sigma^i(j), \sigma^i(2n - j + 1)) = (j + i, 2n + i - j + 1), \]

where the entries in the last transposition must be interpreted modulo \( 2n - 1 \). Suppose that \( \tau_j \) and \( \tau_k \) lie in the same orbit. Then \( \sigma^i\tau_j\sigma^{-i} = \tau_k \) for some \( i \). Thus

\[ (j + i, 2n + i - j + 1) = (k, 2n - k + 1). \]

Since the sum of the two entries in \( \tau_k \) equals \( 2n + 1 \), it follows that \( 2n + 1 + 2i \equiv 2n + 1 \mod (2n - 1) \). Then \( 2i \equiv 0 \mod (2n - 1) \). Therefore, \( i = 0 \) and \( j = k \). \( \square \)
We can now give our second solution to Problem 1(a). Let $\sigma$ and $\tau_1, \ldots, \tau_n$ be as in Lemma 3.2. Let $\theta_1 = \tau_1 \tau_2 \cdots \tau_n$ and let

$$\theta_i = \sigma^{i-1} \theta_1 \sigma^{-(i-1)} = \sigma^{i-1} \tau_1 \tau_2 \cdots \tau_n \sigma^{-(i-1)} = \prod_{j=1}^{n} \sigma^{i-1} \tau_j \sigma^{-(i-1)},$$

$1 \leq i \leq 2n - 1$. Since $\theta_1$ is a product of $n$ pairwise disjoint transpositions, it follows that the same holds for each $\theta_i$, as each is a conjugate of $\theta_1$.

Suppose that $\sigma^{i-1} \tau_j \sigma^{-(i-1)} = \sigma^{m-1} \tau_k \sigma^{-(m-1)}$. Then $j = k$ by Lemma 3.2 because if $j \neq k$, then $\tau_j$ and $\tau_k$ lie in different orbits. Then Lemma 3.1 implies that $i \equiv m \mod 2n - 1$ because the orbit of $\tau_j$ has cardinality $2n - 1$. Thus $i = m$ and we have $n(2n - 1)$ distinct transpositions in these expressions for $\theta_1, \ldots, \theta_{2n-1}$. This again solves Problem 1(a).

**Example 3.3.** If $n = 2$, then $\sigma = (123)$, and this construction gives

$$\begin{align*}
\theta_1 &= (14)(23) \\
\theta_2 &= (24)(13) \\
\theta_3 &= (34)(12).
\end{align*}$$

If $n = 3$, then $\sigma = (12345)$, and this construction gives

$$\begin{align*}
\theta_1 &= (16)(25)(34) \\
\theta_2 &= (26)(13)(45) \\
\theta_3 &= (36)(24)(15) \\
\theta_4 &= (46)(35)(12) \\
\theta_5 &= (56)(14)(23) \\
\theta_6 &= (68)(75)(14)(23) \\
\theta_7 &= (78)(16)(25)(34).
\end{align*}$$

This construction becomes much more transparent if we do not insist that transpositions are written in the form $(ij)$ with $i < j$. We illustrate this now for the case $n = 4$ and $\sigma = (1234567)$.

$$\begin{align*}
\theta_1 &= (18)(27)(36)(45) \\
\theta_2 &= (28)(31)(47)(56) \\
\theta_3 &= (38)(42)(51)(67) \\
\theta_4 &= (48)(53)(62)(71) \\
\theta_5 &= (58)(64)(73)(12) \\
\theta_6 &= (68)(75)(14)(23) \\
\theta_7 &= (78)(16)(25)(34).
\end{align*}$$
One checks that the following rule works. The first row is \( \theta_1 = \tau_1 \tau_2 \cdots \tau_n \). Each succeeding row is obtained by keeping \( 2n \) fixed and adding \( 1 \mod (2n-1) \) to each other entry in the transpositions, noting that \( (2n-1) \) is used instead of \( 0 \).

The solution of Problem 1(c) as given in [H, Theorem 9.1, p.85] gives a 1-factorization of \( K_{2n} \) that corresponds to the solution of Problem 1(a) that we derived from Lemma 3.2.

4. Further problems

**Problem 2:** Which symmetric, constant line-sum matrices with non-negative integer entries can be written as a non-negative linear combination of symmetric permutation matrices?

By a constant line-sum matrix we mean one in which the sum of the entries in each row and in each column are the same. If we don’t require the permutation matrices to be symmetric, then such a sum is always possible. See [L-M] for an exposition of this and related results.

It follows from the solution to Problem 1(b) that the matrix \( J_{2n} \) always has a representation satisfying Problem 2.

The next result gives another case where we can solve Problem 2.

**Proposition 4.1.** For each integer \( m \geq 1 \), there is an equation \( J_m = \sum_{i=1}^{m} P_{\theta_i} \) where each \( P_{\theta_i} \) is a symmetric permutation matrix.

**Proof.** If \( m \) is even this follows from Problem 1(b), using the identity matrix as one of the symmetric permutation matrices.

Now suppose that \( m \) is odd. Since \( m + 1 \) is even, we can write \( J_{m+1} = I_{m+1} + \sum_{i=1}^{m} P'_{\theta_i} \) where each \( P'_{\theta_i} \) is an \((m+1)\times(m+1)\) symmetric permutation matrix. Now define \( m \times m \) matrices \( P_{\theta_i} \) as follows. For \( r \neq s \in \{1, \ldots, m\} \), the \((r,s)\)-entry of \( P_{\theta_i} \) equals the \((r,s)\)-entry of \( P'_{\theta_i} \).

In the last row (and column) of \( P'_{\theta_i} \) there is a 1 in exactly one entry, say in the \( p \)th entry where \( p < m + 1 \). Set the \((p,p)\)-entry of \( P_{\theta_i} \) equal to 1, and all other diagonal entries to 0. See Example 4.2 below for an illustration of this construction.

By construction, the \( P_{\theta_i} \) are all symmetric permutation matrices. Furthermore, since the \( P'_{\theta_i} \) are all disjoint, so are the \( P_{\theta_i} \). \( \square \)

**Example 4.2.** Using Example 2.1, let us write \( J_3 \) in the desired form. We have

\[
\theta_1 = (12)(34), \quad \theta_2 = (13)(24), \quad \theta_3 = (14)(23).
\]

The corresponding matrices are:
$P'_{\theta_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $P'_{\theta_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $P'_{\theta_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

We can obtain a partition for $J_3$ using these matrices by proceeding as in the proof of Proposition 4.1. The resulting matrices are

$P'_{\theta_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $P_{\theta_1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P'_{\theta_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $P_{\theta_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $P'_{\theta_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $P_{\theta_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

We have now seen that several families of symmetric, constant line-sum matrices with non-negative integer entries have a representation satisfying Problem 2. However not every such matrix can be written this way.

**Example 4.3.** Consider the matrix

$B = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$

where $A$ and $C$ are $m \times m$ and $n \times n$ symmetric matrices, respectively, with non-negative integer entries, $m, n$ both odd, and each matrix has every diagonal entry equal to zero. Assume that $B$ is a constant line-sum matrix. This is equivalent to assuming that $A$ and $C$ are each constant line-sum matrices, each with the same line-sum. (For example, we could let $A = C = J_n - I_n$, where $n$ is odd.) Suppose that $B$ has a representation satisfying Problem 2. Then each permutation in the representation would lie either in $\Sigma(\{1, 2, \ldots, m\})$ or in $\Sigma(\{m + 1, m + 2, \ldots, m + n\})$. It follows that each matrix $A$ and $C$ would have a representation satisfying Problem 2. Since $m, n$ are both
odd, any symmetric permutation matrix would have at least one 1 on the diagonal, as it represents a product of disjoint permutations (see Lemma 1.2). This contradicts the assumption that each diagonal entry is zero. Therefore, the matrix \( B \) cannot be written such that the conditions of Problem 2 are met.

Problem 1(a) can be generalized considerably. We content ourselves with the following modest generalization.

**Problem 3:** Let \( S \) be a set of cardinality \( 3n \), \( n \geq 1 \), and let \( T \) denote the set of three-element subsets of \( S \). Then \( T \) has cardinality

\[
\binom{3n}{3} = \frac{3n(3n-1)(3n-2)}{6} = n \frac{(3n-1)(3n-2)}{2} = n \binom{3n-1}{2}.
\]

Let \( m = \binom{3n-1}{2} \). Can one partition \( T \) into \( m \) subsets \( T_1, \ldots, T_m \) such that each \( T_i \) has cardinality \( n \), and for each \( i \), \( 1 \leq i \leq m \), each element of \( S \) occurs exactly once in the three-element subsets in \( T_i \)?

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