**Fractal Geometry**

A Fractal is a geometric object whose dimension is fractional. Most fractals are self similar, that is when any small part of a fractal is magnified the result resembles the original fractal. Examples of fractals are the Koch curve, the Seirpinski gasket, and the Menger sponge.

**Similarity Dimension:**

- Take a line segment of length one unit and divide it into $N$ equal parts. let $S = \frac{\text{old length}}{\text{new length}}$, then $N = S^1$.

  $N = 3, S = 3$

- Take a square (dimension 2) of area one square unit and divide it into $N$ congruent parts. Then $S = \sqrt{N}$ and $N = S^2$.

  $N = 4, S = 2$

- Take a cube (dimension 3) of volume one cubic unit and divide it into $N$ congruent parts. Then $S = N^{\frac{1}{3}}$ and $N = S^3$.

  $N = 8, S = 2$
So given set of dimension $d$ consisting of $N$ parts congruent part then $N = S^d$, where $S = \frac{\text{old length}}{\text{new length}}$. Solving for $d$ yields

$$d = \frac{\ln N}{\ln S}$$

**Exercise 1:**

$T_\infty = \lim_{n \to \infty} T_n$ is called the **Seirpinski Triangle or Gasket**.

1. What is the perimeter of $T_n$?
2. What is the area of $T_n$?
3. Compute the fractal dimension of $T_\infty$.

**Solution:**

1. $P_n = \left(\frac{3}{2}\right)^n P_0$.
2. $A_n = \left(\frac{3}{4}\right)^n A_0$.
3. $N = 3$ and $S = 2$, hence $d = \frac{\ln 3}{\ln 2}$. 
Exercise 2:

$C_\infty = \lim_{n \to \infty} C_n$ is called the Seirpinski Carpet.

1. What is the perimeter of $C_n$?
2. What is the area of $C_n$?
3. Compute the fractal dimension of $C_\infty$.

Solution:

1. $P_n = P_0 + \frac{1}{3} P_0 + \frac{8}{3^2} P_0 + \cdots + \frac{8^{n-1}}{3^n} P_0$.
2. $A_n = (\frac{8}{9})^n A_0$.
3. $N = 8$ and $S = 3$, hence $d = \frac{\ln 8}{\ln 3}$. 
Exercise 3:

\( M_\infty = \lim_{n \to \infty} M_n \) is called the Menger Sponge.

1. What is the volume of \( M_n \).
2. What is the surface area of \( M_n \)?
3. Compute the fractal dimension of \( M_\infty \).

Solution:

1. \( V_0 = 2^3 = 8 \) and \( V_n = 20 \cdot (1/3)^3 \cdot (V_{n-1}) \).
2. \( A_0 = 6 \cdot 2^2 = 24 \) and \( A_n = 20 \cdot (1/3)^2 \cdot (A_{n-1}) - 24 \cdot (1/3)^2 \cdot (A_{t_{n-1}}) \cdot 2 \)
   \( = (20/9) \cdot \text{Area}(C_{n-1}) - (64/3) \cdot (8/9)^{n-1} \) where \( \text{Area}(C_{n-1}) \) is
   the area of the \( n - 1 \) stage in the construction of the Sierpinski carpet. Hence \( A_n = (8/9)^n \cdot 16 + (20/9)^n \cdot 8 \).
3. \( N = 20 \) and \( S = 3 \), hence \( d = \frac{\ln 20}{\ln 3} \).
**Exercise 4:**

Draw a fractal with dimension $\frac{3}{2}$.

**Solution:**

![Fractal Diagram](attachment:fractal.png)

Demonstrate the applet “Fractal Dimension” at:

Box Dimension:
Box dimension is another way to measure fractal dimension, it is defined as follows: If $X$ is a bounded subset of the Euclidean space, and $\epsilon \geq 0$. Let $N(X, \epsilon)$ be the minimal number of boxes in the grid of side length $\epsilon$ which are required to cover $X$. We say that $X$ has box dimension $D$ if the following limit exists and has value $D$.

$$\lim_{\epsilon \to 0^+} \frac{\ln(N(X, \epsilon))}{\ln(\frac{1}{\epsilon})}$$

Consider for example the Seirpinski gasket:

\[
\begin{array}{c|c}
\epsilon & N(X, \epsilon) \\
\hline
1 & 1 \\
\frac{1}{2} & 3 \\
\hline
\frac{1}{4} & 9 \\
\vdots & \vdots \\
\frac{1}{2^n} & 3^n \\
\end{array}
\]

Hence

$$D = \lim_{n \to \infty} \frac{\ln(4 \cdot 3^n)}{\ln(2^n)} = \frac{\ln 3}{\ln 2}$$
Demonstrate the applet “Box Counting Dimension” at:

http://classes.yale.edu/fractals/Software/boxdim.html

Geometrically, if you plot the results on a graph with \( \ln N(X, \epsilon) \) on the vertical axis and \( \ln(1/\epsilon) \) on the horizontal axis, then the slope of the best fit line of the data will be an approximation of Box dimension of the fractal.
Discuss computing the fractal dimension of a bunch of broccoli.


My Results:

<table>
<thead>
<tr>
<th>Ball Size</th>
<th>N</th>
<th>$\frac{1}{\epsilon}$</th>
<th>$\ln(\frac{1}{\epsilon})$</th>
<th>$\ln(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6x5x2.5</td>
<td>10</td>
<td>1/4.5</td>
<td>-1.504</td>
<td>2.303</td>
</tr>
<tr>
<td>4x4x2.5</td>
<td>19</td>
<td>1/3.5</td>
<td>-1.253</td>
<td>2.94</td>
</tr>
<tr>
<td>3.5x3x2</td>
<td>28</td>
<td>1/2.83</td>
<td>-1.04</td>
<td>3.33</td>
</tr>
<tr>
<td>3x2.5x2</td>
<td>41</td>
<td>1/2.5</td>
<td>-.916</td>
<td>3.71</td>
</tr>
<tr>
<td>2.5x2.5x2</td>
<td>64</td>
<td>1/2.33</td>
<td>-.847</td>
<td>4.158</td>
</tr>
</tbody>
</table>

Use the linear regression applet at http://math.hws.edu/javamath/ryan/Regression.html to plot the data points and find the slope of the best fit line.

Fractal dimension = slope = 2.62.

If you throw the 1st point out you get fractal dimension = slope = 2.82.
**Divider Dimension:**

A fractal curve has fractal (or divider) dimension $D$ if its length $L$ when measured with rods of length $l$ is given by

$$L = C \cdot l^{1-D}$$

$C$ is a constant that is a certain measure of the apparent length, and the equation above must be true for several different values of $l$. Consider the Koch curve:

![Koch Curve Diagram]

<table>
<thead>
<tr>
<th>$l$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{1}{3}$</td>
<td>$\frac{4}{3}$</td>
</tr>
<tr>
<td>$\frac{1}{9}$</td>
<td>$\frac{16}{9}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\frac{1}{3^n}$</td>
<td>$(\frac{4}{3})^n$</td>
</tr>
</tbody>
</table>

It appears to have length $L = (\frac{4}{3})^n$ when measured with rods of length $l = (\frac{1}{3})^n$.

Hence, $(\frac{4}{3})^n = C \cdot (\frac{1}{3})^{n(1-D)}$

This implies that $n \ln \frac{4}{3} = \ln(C) + n(1 - D) \ln \frac{1}{3}$.

In order for the equation to be satisfied for all $n$, $\ln C$ must be zero, hence $C = 1$ and so $D = \frac{\ln 4}{\ln 3}$.

Geometrically, if you graph $\ln(L)$ versus $\ln(l)$ the $D = 1$—Slope of line.
**Exercise 5:**

The west coast of Britain when measured with rods of length 10 km is 3020 km. But when measured with rods of length 100 km it is 1700 km. What is the fractal dimension of the west coast of Britain?

Demonstrate the applet “Coastline of Hong Kong” at:

**Affine Transformations:**

An affine transformation is of the form

\[
f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}
\]

or

\[
f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} r \cos(\theta) & -s \sin(\phi) \\ r \sin(\theta) & s \cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}
\]

Where \( r \) and \( s \) are the scaling factors in the \( x \) and \( y \) directions respectively. \( \theta \) and \( \phi \) measure rotation of horizontal and vertical lines resp. \( e \) and \( f \) measure horizontal and vertical translations resp.

It can be easily shown that \( r^2 = a^2 + c^2 \), \( s^2 = b^2 + d^2 \), \( \theta = \arctan(c/a) \) and \( \phi = \arctan(-b/d) \).
A reflection about the y-axis would be given by
\[ x' = -x \quad \text{and} \quad y' = y \]
---
matrix form \[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

A reflection about the x-axis would be given by
\[ x' = x \quad \text{and} \quad y' = -y \]
---
matrix form \[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

**Rotation is represented as follows:**

Matrix form: 
\[ \begin{pmatrix} \cos(\theta) & -\sin(\phi) \\ \sin(\theta) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]

\( x' = \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \)
\( y' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \)
Shears are represented as follows:

Shear in the y direction

\[ x' = x \]
\[ y' = 2x + y \]

Shear in the x direction

\[ x' = x + 2y \]
\[ y' = y \]
Iterated Function Systems:
An Iterated Function System in $\mathbb{R}^2$ is a collection $\{F_1, F_2, \ldots, F_n\}$ of contraction mappings on $\mathbb{R}^2$. ($F_i$ is a contraction mapping if given $x, y$ then $d(F_i(x), F_i(y)) \leq sd(x, y)$ where $0 \leq s < 1$)

The Collage Theorem says that there is a unique nonempty compact subset $A \subset \mathbb{R}^2$ called the attractor such that

$$A = F_1(A) \cup F_2(A) \cup \ldots \cup F_n(A)$$

Sketch of Proof:
Given a complete metric space $X$, let $H(X)$ be the collection of nonempty compact subsets of $X$. Given $A$ and $B$ in $H(X)$ define $d(A, B)$ to be the smallest number $r$ such that each point of in $A$ is within $r$ of some point in $B$ and vice versa. If $X$ is complete so is $H(X)$.

Given an IFS on $X$, define $G : H(X) \to H(X)$ by $G(K) = F_1(K) \cup \ldots \cup F_n(K)$. If each $F_i$ is a contraction then so is $G$. The contraction mapping theorem says that a contraction mapping on a complete space has a unique fixed point. Hence there is $A \in H(X)$ such that $G(A) = A = F_1(A) \cup F_2(A) \cup \ldots \cup F_n(A)$.

Given $K_0 \subset \mathbb{R}^2$, let $K_n = G^n(K_0)$. Then

$$d(A, K_i) \leq \frac{s^i d(K_0, G(K_0))}{1 - s}$$

Hence $K_i$ is a very good approximation of $A$ for large enough $i$.

Deterministic Algorithm:  (Slow!)
Given $K_0$, let $K_n = G^n(K_0)$. Now $K_n \to A$.

Random Algorithm:  (Fast!)
Choose $P_0 \in K_0$ and let $P_{i+1} = F_j(P_i)$ where $F_j$ is chosen at random from all of the $F_i$. Clearly $P_i \in K_i$ and for sufficiently large $i$, $P_i$ is arbitrarily close to some point in $A$. 

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Finding IFS Rules from Images of Points

Given three non-collinear initial points \(p_1 = (x_1, y_1), p_2 = (x_2, y_2), p_3 = (x_3, y_3)\) and three image points \(q_1 = (u_1, v_1), q_2 = (u_2, v_2), q_3 = (u_3, v_3)\) respectively. Find an affine transformation \(T\) defined by

\[
T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}
\]

such that \(T(p_1) = q_1, T(p_2) = q_2, \) and \(T(p_3) = q_3.\)

This gives the following six equations in six unknowns:

\[
\begin{align*}
ax_1 + by_1 + e &= u_1 \\
cx_1 + dy_1 + f &= v_1 \\
ax_2 + by_2 + e &= u_2 \\
cx_2 + dy_2 + f &= v_2 \\
ax_3 + by_3 + e &= u_3 \\
cx_3 + dy_3 + f &= v_3
\end{align*}
\]

This system has a unique solution if and only if the points \(p_1, p_2, \) and \(p_3\) are noncollinear.
The IFS for the Sierpinski triangle is \( \{T_1, T_2, T_3\} \) where

\[
T_1\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
T_2\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ .5 \end{pmatrix}
\]

\[
T_3\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} .5 & 0 \\ 0 & .5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ .5 \end{pmatrix}
\]
Example:

The IFS for the Seirpinski carpet is \( \{ T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8 \} \) where

\[
T_1\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ 0 \end{array}\right) \\
T_2\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} .333 \\ 0 \end{array}\right) \\
T_3\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} .666 \\ 0 \end{array}\right) \\
T_4\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ .333 \end{array}\right) \\
T_5\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} .666 \\ .333 \end{array}\right) \\
T_6\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ .666 \end{array}\right) \\
T_7\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} .333 \\ .666 \end{array}\right) \\
T_8\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} .333 & 0 \\ 0 & .333 \end{array}\right) \cdot \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} .666 \\ .666 \end{array}\right)
\]
Example:

The IFS for the tree is \( \{ T_1, T_2, T_3 \} \) where

\[
\begin{align*}
T_1 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) &= \left( \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 15 \end{pmatrix} \\
T_2 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) &= \left( \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix} \\
T_3 \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) &= \left( \begin{pmatrix} 0 & 0 \\ 0 & 0.333 \end{pmatrix} \right) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \end{pmatrix}
\end{align*}
\]
Download and experiment with the freeware “IFS Construction Kit” which can be used to design and draw fractals based on iterated function systems. You can find it at:

http://ecademy.agnesscott.edu/~lriddle/jfskit/

• Discuss the IFS for the Fern.
• Discuss the IFS for the Tree.
• Discuss the IFS for the mountain range.
• Discuss the IFS for the Wave.
Mandelbrot and Julia Sets:
Consider the complex function \( F_c(z) = z^2 + c \). Given an initial point \( z_0 \) define its orbit under \( F_c \) to be \( z_0, z_1, z_2, \ldots \) where \( z_{n+1} = z_n^2 + c \). Define the escape set, \( E_c \), and the prisoner set, \( P_c \), for for the parameter \( c \) to be:

\[
E_c = \{ z_0 : |z_n| \to \infty \text{ as } n \to \infty \}, \quad \text{and} \quad E_c = \{ z_0 | z_0 \notin E_c \}
\]

Now the Julia set for the parameter \( c \), \( J_c \), is the boundary of the escape set \( E_c \).

For example, if \( c = 0 \) then \( F_c(z) = z^2 \). Now if \( z = re^{i\theta} \) then \( F(z) = r^2e^{2i\theta} \).

|   | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) |
|---|---|---|---|---|---|---|
| \( z \) | .8 | 15 | | | 1 | 15 |
| \( z^2 \) | .64 | 30 | | | 1 | 30 |
| \( z^4 \) | .4096 | 60 | | | 1 | 60 |
| \( z^8 \) | .1678 | 120 | | | 1 | 120 |
| \( z^{16} \) | .0281 | 240 | | | 1 | 240 |
| \( z^{32} \) | .0008 | 120 | | | 1 | 120 |

|   | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) |
|---|---|---|---|---|---|---|
| \( z_0 \) | < 1 | | \( z_1 \) | | | 1 |
| \( z_1 \) | | | \( z_2 \) | | | |
| \( z_2 \) | | | \( z_3 \) | | | |
| \( z_3 \) | | | \( z_4 \) | | | |
| \( z_4 \) | | | | | | |
| \( z_5 \) | | | | | | |
| ... | | | | | | |

|   | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) | \( |z_0| \) | \( \theta \) |
|---|---|---|---|---|---|---|
| \( z_0 \) | = 1 | | \( z_1 \) | | | |
| \( z_1 \) | | | \( z_2 \) | | | |
| \( z_2 \) | | | \( z_3 \) | | | |
| \( z_3 \) | | | \( z_4 \) | | | |
| \( z_4 \) | | | | | | |
| \( z_5 \) | | | | | | |
| ... | | | | | | |

20
Hence \( E_c = \{ z : |z| > 1 \} \), \( P_c = \{ z : |z| \leq 1 \} \), and therefore \( J_0 \) is the unit circle.

The Julia set is either connected (one piece) or totally disconnected (dust).

Now we define the Mandelbrot set to be

\[
M = \{ c \in \mathbb{C} : J_c \text{ is connected} \}
\]

Experiment with the Java applet “Mandelbrot and Julia Sets” at:

[http://classes.yale.edu/fractals/](http://classes.yale.edu/fractals/)

Download “winfeed” at the address below and use it to plot the Julia sets for the constants \( c = -0.5 + 0.5i \), \( c = -0.7454285 + 0.1130089i \), and others.

[http://math.exeter.edu/rparris/winfeed.html](http://math.exeter.edu/rparris/winfeed.html)