*Chapter Five*

# Similarity

While studying matrix equivalence, we have shown that for any homomorphism there are bases $B$ and $D$ such that the representation matrix has a block partial-identity form.

$$\operatorname{Rep}_{B,D}(h) = \left(\begin{array}{c|c} \textit{Identity} & \textit{Zero} \\ \hline \textit{Zero} & \textit{Zero} \end{array}\right)$$

This representation describes the map as sending $c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ to $c_1\vec{\delta}_1 + \cdots + c_k\vec{\delta}_k + \vec{0} + \cdots + \vec{0}$, where $n$ is the dimension of the domain and $k$ is the dimension of the range. So, under this representation the action of the map is easy to understand because most of the matrix entries are zero.

This chapter considers the special case where the domain and the codomain are equal, that is, where the homomorphism is a transformation. In this case we naturally ask to find a single basis $B$ so that $\operatorname{Rep}_{B,B}(t)$ is as simple as possible (we will take 'simple' to mean that it has many zeroes). A matrix having the above block partial-identity form is not always possible here. But we will develop a form that comes close, a representation that is nearly diagonal.

## I Complex Vector Spaces

This chapter requires that we factor polynomials. Of course, many polynomials do not factor over the real numbers; for instance, $x^2 + 1$ does not factor into the product of two linear polynomials with real coefficients. For that reason, we shall from now on take our scalars from the complex numbers.

That is, we are shifting from studying vector spaces over the real numbers to vector spaces over the complex numbers — in this chapter vector and matrix entries are complex.

Any real number is a complex number and a glance through this chapter shows that most of the examples use only real numbers. Nonetheless, the critical theorems require that the scalars be complex numbers, so the first section below is a quick review of complex numbers.

In this book we are moving to the more general context of taking scalars to be complex only for the pragmatic reason that we must do so in order to develop the representation. We will not go into using other sets of scalars in more detail because it could distract from our goal. However, the idea of taking scalars from a structure other than the real numbers is an interesting one. Delightful presentations taking this approach are in [Halmos] and [Hoffman & Kunze].

## I.1 Factoring and Complex Numbers; A Review

*This subsection is a review only and we take the main results as known. For proofs, see [Birkhoff & MacLane] or [Ebbinghaus].*

Just as integers have a division operation — e.g., '4 goes 5 times into 21 with remainder 1' — so do polynomials.

**1.1 Theorem (Division Theorem for Polynomials)** Let $c(x)$ be a polynomial. If $m(x)$ is a non-zero polynomial then there are *quotient* and *remainder* polynomials $q(x)$ and $r(x)$ such that

$$c(x) = m(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is strictly less than the degree of $m(x)$.

In this book constant polynomials, including the zero polynomial, are said to have degree 0. (This is not the standard definition, but it is convienent here.)

The point of the integer division statement '4 goes 5 times into 21 with remainder 1' is that the remainder is less than 4 — while 4 goes 5 times, it does not go 6 times. In the same way, the point of the polynomial division statement is its final clause.

**1.2 Example** If $c(x) = 2x^3 - 3x^2 + 4x$ and $m(x) = x^2 + 1$ then $q(x) = 2x - 3$ and $r(x) = 2x + 3$. Note that $r(x)$ has a lower degree than $m(x)$.

**1.3 Corollary** The remainder when $c(x)$ is divided by $x - \lambda$ is the constant polynomial $r(x) = c(\lambda)$.

Proof. The remainder must be a constant polynomial because it is of degree less than the divisor $x - \lambda$, To determine the constant, take $m(x)$ from the theorem to be $x - \lambda$ and substitute $\lambda$ for $x$ to get $c(\lambda) = (\lambda - \lambda) \cdot q(\lambda) + r(x)$. QED

If a divisor $m(x)$ goes into a dividend $c(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $m(x)$ is a *factor* of $c(x)$. Any *root* of the factor (any $\lambda \in \mathbb{R}$ such that $m(\lambda) = 0$) is a root of $c(x)$ since $c(\lambda) = m(\lambda) \cdot q(\lambda) = 0$. The prior corollary immediately yields the following converse.

**1.4 Corollary** If $\lambda$ is a root of the polynomial $c(x)$ then $x - \lambda$ divides $c(x)$ evenly, that is, $x - \lambda$ is a factor of $c(x)$.

Finding the roots and factors of a high-degree polynomial can be hard. But for second-degree polynomials we have the quadratic formula: the roots of $ax^2 + bx + c$ are

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(if the discriminant $b^2 - 4ac$ is negative then the polynomial has no real number roots). A polynomial that cannot be factored into two lower-degree polynomials with real number coefficients is *irreducible over the reals*.

**1.5 Theorem** Any constant or linear polynomial is irreducible over the reals. A quadratic polynomial is irreducible over the reals if and only if its discriminant is negative. No cubic or higher-degree polynomial is irreducible over the reals.

**1.6 Corollary** Any polynomial with real coefficients can be factored into linear and irreducible quadratic polynomials. That factorization is unique; any two factorizations have the same powers of the same factors.

Note the analogy with the prime factorization of integers. In both cases, the uniqueness clause is very useful.

**1.7 Example** Because of uniqueness we know, without multiplying them out, that $(x+3)^2(x^2+1)^3$ does not equal $(x+3)^4(x^2+x+1)^2$.

**1.8 Example** By uniqueness, if $c(x) = m(x) \cdot q(x)$ then where $c(x) = (x-3)^2(x+2)^3$ and $m(x) = (x-3)(x+2)^2$, we know that $q(x) = (x-3)(x+2)$.

While $x^2 + 1$ has no real roots and so doesn't factor over the real numbers, if we imagine a root — traditionally denoted $i$ so that $i^2 + 1 = 0$ — then $x^2 + 1$ factors into a product of linears $(x - i)(x + i)$.

So we adjoin this root $i$ to the reals and close the new system with respect to addition, multiplication, etc. (i.e., we also add $3 + i$, and $2i$, and $3 + 2i$, etc., putting in all linear combinations of 1 and $i$). We then get a new structure, the *complex numbers*, denoted $\mathbb{C}$.

In $\mathbb{C}$ we can factor (obviously, at least some) quadratics that would be irreducible if we were to stick to the real numbers. Surprisingly, in $\mathbb{C}$ we can not only factor $x^2 + 1$ and its close relatives, we can factor any quadratic.

$$ax^2 + bx + c = a \cdot \big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\big) \cdot \big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\big)$$

**1.9 Example** The second degree polynomial $x^2+x+1$ factors over the complex numbers into the product of two first degree polynomials.

$$\big(x - \frac{-1 + \sqrt{-3}}{2}\big)\big(x - \frac{-1 - \sqrt{-3}}{2}\big) = \big(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\big)\big(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)\big)$$

**1.10 Corollary (Fundamental Theorem of Algebra)** Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

## I.2 Complex Representations

Recall the definitions of the complex number addition

$$(a+bi)\ +\ (c+di) = (a+c)+(b+d)i$$

and multiplication.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bd(-1)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

**2.1 Example** For instance, $(1-2i)\ +\ (5+4i) = 6+2i$ and $(2-3i)(4-0.5i) = 6.5-13i$.

Handling scalar operations with those rules, all of the operations that we've covered for real vector spaces carry over unchanged.

**2.2 Example** Matrix multiplication is the same, although the scalar arithmetic involves more bookkeeping.

$$\begin{aligned}&\begin{pmatrix}1+1i & 2-0i\\ i & -2+3i\end{pmatrix}\begin{pmatrix}1+0i & 1-0i\\ 3i & -i\end{pmatrix}\\ &=\begin{pmatrix}(1+1i)\cdot(1+0i)+(2-0i)\cdot(3i) & (1+1i)\cdot(1-0i)+(2-0i)\cdot(-i)\\ (i)\cdot(1+0i)+(-2+3i)\cdot(3i) & (i)\cdot(1-0i)+(-2+3i)\cdot(-i)\end{pmatrix}\\ &=\begin{pmatrix}1+7i & 1-1i\\ -9-5i & 3+3i\end{pmatrix}\end{aligned}$$

Everything else from prior chapters that we can, we shall also carry over unchanged. For instance, we shall call this

$$\langle\begin{pmatrix}1+0i\\ 0+0i\\ \vdots\\ 0+0i\end{pmatrix},\dots,\begin{pmatrix}0+0i\\ 0+0i\\ \vdots\\ 1+0i\end{pmatrix}\rangle$$

the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and again denote it $\mathcal{E}_n$.

# II Similarity

## II.1 Definition and Examples

We've defined $H$ and $\hat{H}$ to be matrix-equivalent if there are nonsingular matrices $P$ and $Q$ such that $\hat{H} = PHQ$. That definition is motivated by this diagram

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow[H]{h} & W_{\text{w.r.t. } D} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ V_{\text{w.r.t. } \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\text{w.r.t. } \hat{D}} \end{array}$$

showing that $H$ and $\hat{H}$ both represent $h$ but with respect to different pairs of bases. We now specialize that setup to the case where the codomain equals the domain, and where the codomain's basis equals the domain's basis.

$$\begin{array}{ccc} V_{\text{w.r.t. } B} & \xrightarrow{t} & V_{\text{w.r.t. } B} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ V_{\text{w.r.t. } D} & \xrightarrow{t} & V_{\text{w.r.t. } D} \end{array}$$

To move from the lower left to the lower right we can either go straight over, or up, over, and then down. In matrix terms,

$$\text{Rep}_{D,D}(t) = \text{Rep}_{B,D}(\text{id})\,\text{Rep}_{B,B}(t)\,\big(\text{Rep}_{B,D}(\text{id})\big)^{-1}$$

(recall that a representation of composition like this one reads right to left).

**1.1 Definition** The matrices $T$ and $S$ are *similar* if there is a nonsingular $P$ such that $T = PSP^{-1}$.

Since nonsingular matrices are square, the similar matrices $T$ and $S$ must be square and of the same size.

**1.2 Example** With these two,

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

calculation gives that $S$ is similar to this matrix.

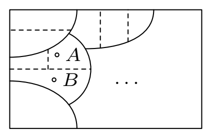
$$T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

**1.3 Example** The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The only matrix similar to the identity matrix is itself: $PIP^{-1} = PP^{-1} = I$.

Since matrix similarity is a special case of matrix equivalence, if two matrices are similar then they are equivalent. What about the converse: must matrix equivalent square matrices be similar? The answer is no. The prior example shows that the similarity classes are different from the matrix equivalence classes, because the matrix equivalence class of the identity consists of all nonsingular matrices of that size. Thus, for instance, these two are matrix equivalent but not similar.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

So some matrix equivalence classes split into two or more similarity classes — similarity gives a finer partition than does equivalence. This picture shows some matrix equivalence classes subdivided into similarity classes.




To understand the similarity relation we shall study the similarity classes. We approach this question in the same way that we've studied both the row equivalence and matrix equivalence relations, by finding a canonical form for representatives* of the similarity classes, called Jordan form. With this canonical form, we can decide if two matrices are similar by checking whether they reduce to the same representative. We've also seen with both row equivalence and matrix equivalence that a canonical form gives us insight into the ways in which members of the same class are alike (e.g., two identically-sized matrices are matrix equivalent if and only if they have the same rank).

## Exercises

**1.4** For

$$S = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 \\ -11/2 & -5 \end{pmatrix} \quad P = \begin{pmatrix} 4 & 2 \\ -3 & 2 \end{pmatrix}$$

check that $T = PSP^{-1}$.

✓ **1.5** Example 1.3 shows that the only matrix similar to a zero matrix is itself and that the only matrix similar to the identity is itself.

**(a)** Show that the $1\times1$ matrix $(2)$, also, is similar only to itself.

**(b)** Is a matrix of the form $cI$ for some scalar $c$ similar only to itself?

**(c)** Is a diagonal matrix similar only to itself?

**1.6** Show that these matrices are not similar.

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

* More information on representatives is in the appendix.

**1.7** Consider the transformation $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ described by $x^2 \mapsto x+1$, $x \mapsto x^2-1$, and $1 \mapsto 3$.

**(a)** Find $T = \mathrm{Rep}_{B,B}(t)$ where $B = \langle x^2, x, 1\rangle$.

**(b)** Find $S = \mathrm{Rep}_{D,D}(t)$ where $D = \langle 1, 1+x, 1+x+x^2\rangle$.

**(c)** Find the matrix $P$ such that $T = PSP^{-1}$.

✓ **1.8** Exhibit an nontrivial similarity relationship in this way: let $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ act by

$$\begin{pmatrix}1\\2\end{pmatrix} \mapsto \begin{pmatrix}3\\0\end{pmatrix} \qquad \begin{pmatrix}-1\\1\end{pmatrix} \mapsto \begin{pmatrix}-1\\2\end{pmatrix}$$

and pick two bases, and represent $t$ with respect to then $T = \mathrm{Rep}_{B,B}(t)$ and $S = \mathrm{Rep}_{D,D}(t)$. Then compute the $P$ and $P^{-1}$ to change bases from $B$ to $D$ and back again.

**1.9** Explain Example 1.3 in terms of maps.

✓ **1.10** Are there two matrices $A$ and $B$ that are similar while $A^2$ and $B^2$ are not similar? [Halmos]

✓ **1.11** Prove that if two matrices are similar and one is invertible then so is the other.

✓ **1.12** Show that similarity is an equivalence relation.

**1.13** Consider a matrix representing, with respect to some $B, B$, reflection across the $x$-axis in $\mathbb{R}^2$. Consider also a matrix representing, with respect to some $D, D$, reflection across the $y$-axis. Must they be similar?

**1.14** Prove that similarity preserves determinants and rank. Does the converse hold?

**1.15** Is there a matrix equivalence class with only one matrix similarity class inside? One with infinitely many similarity classes?

**1.16** Can two different diagonal matrices be in the same similarity class?

✓ **1.17** Prove that if two matrices are similar then their $k$-th powers are similar when $k > 0$. What if $k \leq 0$?

✓ **1.18** Let $p(x)$ be the polynomial $c_n x^n + \cdots + c_1 x + c_0$. Show that if $T$ is similar to $S$ then $p(T) = c_n T^n + \cdots + c_1 T + c_0 I$ is similar to $p(S) = c_n S^n + \cdots + c_1 S + c_0 I$.

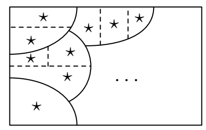
**1.19** List all of the matrix equivalence classes of $1\times 1$ matrices. Also list the similarity classes, and describe which similarity classes are contained inside of each matrix equivalence class.

**1.20** Does similarity preserve sums?

**1.21** Show that if $T - \lambda I$ and $N$ are similar matrices then $T$ and $N + \lambda I$ are also similar.

## II.2 Diagonalizability

The prior subsection defines the relation of similarity and shows that, although similar matrices are necessarily matrix equivalent, the converse does not hold. Some matrix-equivalence classes break into two or more similarity classes (the nonsingular $n\times n$ matrices, for instance). This means that the canonical form for matrix equivalence, a block partial-identity, cannot be used as a canonical form for matrix similarity because the partial-identities cannot be in more than one similarity class, so there are similarity classes without one. This picture illustrates. As earlier in this book, class representatives are shown with stars.




We are developing a canonical form for representatives of the similarity classes. We naturally try to build on our previous work, meaning first that the partial identity matrices should represent the similarity classes into which they fall, and beyond that, that the representatives should be as simple as possible. The simplest extension of the partial-identity form is a diagonal form.

**2.1 Definition** A transformation is *diagonalizable* if it has a diagonal representation with respect to the same basis for the codomain as for the domain. A *diagonalizable matrix* is one that is similar to a diagonal matrix: $T$ is diagonalizable if there is a nonsingular $P$ such that $PTP^{-1}$ is diagonal.

**2.2 Example** The matrix

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

is diagonalizable.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

**2.3 Example** Not every matrix is diagonalizable. The square of

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is the zero matrix. Thus, for any map $n$ that $N$ represents (with respect to the same basis for the domain as for the codomain), the composition $n \circ n$ is the zero map. This implies that no such map $n$ can be diagonally represented (with respect to any $B, B$) because no power of a nonzero diagonal matrix is zero. That is, there is no diagonal matrix in $N$'s similarity class.

That example shows that a diagonal form will not do for a canonical form — we cannot find a diagonal matrix in each matrix similarity class. However, the canonical form that we are developing has the property that if a matrix can be diagonalized then the diagonal matrix is the canonical representative of the similarity class. The next result characterizes which maps can be diagonalized.

**2.4 Corollary** A transformation $t$ is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each $i$.

Proof. This follows from the definition by considering a diagonal representation matrix.

$$\mathrm{Rep}_{B,B}(t)=\left(\begin{array}{c|c|c} \vdots & & \vdots \\ \mathrm{Rep}_B(t(\vec{\beta}_1)) & \cdots & \mathrm{Rep}_B(t(\vec{\beta}_n)) \\ \vdots & & \vdots \end{array}\right)=\left(\begin{array}{c|c|c} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{array}\right)$$

This representation is equivalent to the existence of a basis satisfying the stated conditions simply by the definition of matrix representation. QED

**2.5 Example** To diagonalize

$$T=\begin{pmatrix}3&2\\0&1\end{pmatrix}$$

we take it as the representation of a transformation with respect to the standard basis $T=\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ and we look for a basis $B=\langle\vec{\beta}_1,\vec{\beta}_2\rangle$ such that

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}\lambda_1&0\\0&\lambda_2\end{pmatrix}$$

that is, such that $t(\vec{\beta}_1)=\lambda_1\vec{\beta}_1$ and $t(\vec{\beta}_2)=\lambda_2\vec{\beta}_2$.

$$\begin{pmatrix}3&2\\0&1\end{pmatrix}\vec{\beta}_1=\lambda_1\cdot\vec{\beta}_1 \qquad \begin{pmatrix}3&2\\0&1\end{pmatrix}\vec{\beta}_2=\lambda_2\cdot\vec{\beta}_2$$

We are looking for scalars $x$ such that this equation

$$\begin{pmatrix}3&2\\0&1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}=x\cdot\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

has solutions $b_1$ and $b_2$, which are not both zero. Rewrite that as a linear system.

$$\begin{array}{rcr}(3-x)\cdot b_1+&2\cdot b_2&=0\\&(1-x)\cdot b_2&=0\end{array}\qquad(*)$$

In the bottom equation the two numbers multiply to give zero only if at least one of them is zero so there are two possibilities, $b_2=0$ and $x=1$. In the $b_2=0$ possibility, the first equation gives that either $b_1=0$ or $x=3$. Since the case of both $b_1=0$ and $b_2=0$ is disallowed, we are left looking at the possibility of $x=3$. With it, the first equation in $(*)$ is $0\cdot b_1+2\cdot b_2=0$ and so associated with 3 are vectors with a second component of zero and a first component that is free.

$$\begin{pmatrix}3&2\\0&1\end{pmatrix}\begin{pmatrix}b_1\\0\end{pmatrix}=3\cdot\begin{pmatrix}b_1\\0\end{pmatrix}$$

That is, one solution to $(*)$ is $\lambda_1=3$, and we have a first basis vector.

$$\vec{\beta}_1=\begin{pmatrix}1\\0\end{pmatrix}$$

In the $x = 1$ possibility, the first equation in $(*)$ is $2 \cdot b_1 + 2 \cdot b_2 = 0$, and so associated with 1 are vectors whose second component is the negative of their first component.

$$\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix} = 1 \cdot \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix}$$

Thus, another solution is $\lambda_2 = 1$ and a second basis vector is this.

$$\vec{\beta}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To finish, drawing the similarity diagram

$$\begin{array}{ccc} \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow & & \text{id}\big\downarrow \\ \mathbb{R}^2_{\text{w.r.t. } B} & \xrightarrow[D]{t} & \mathbb{R}^2_{\text{w.r.t. } B} \end{array}$$

and noting that the matrix $\text{Rep}_{B,\mathcal{E}_2}(\text{id})$ is easy leads to this diagonalization.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

In the next subsection, we will expand on that example by considering more closely the property of Corollary 2.4. This includes seeing another way, the way that we will routinely use, to find the $\lambda$'s.

## Exercises

✓ **2.6** Repeat Example 2.5 for the matrix from Example 2.2.

**2.7** Diagonalize these upper triangular matrices.

**(a)** $\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$

✓ **2.8** What form do the powers of a diagonal matrix have?

**2.9** Give two same-sized diagonal matrices that are not similar. Must any two different diagonal matrices come from different similarity classes?

**2.10** Give a nonsingular diagonal matrix. Can a diagonal matrix ever be singular?

✓ **2.11** Show that the inverse of a diagonal matrix is the diagonal of the the inverses, if no element on that diagonal is zero. What happens when a diagonal entry is zero?

**2.12** The equation ending Example 2.5

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

is a bit jarring because for $P$ we must take the first matrix, which is shown as an inverse, and for $P^{-1}$ we take the inverse of the first matrix, so that the two $-1$ powers cancel and this matrix is shown without a superscript $-1$.

**(a)** Check that this nicer-appearing equation holds.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}$$

**(b)** Is the previous item a coincidence? Or can we always switch the $P$ and the $P^{-1}$?

**2.13** Show that the $P$ used to diagonalize in Example 2.5 is not unique.

**2.14** Find a formula for the powers of this matrix *Hint*: see Exercise 8.

$$\begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix}$$

✓ **2.15** Diagonalize these.

**(a)** $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

**2.16** We can ask how diagonalization interacts with the matrix operations. Assume that $t, s\colon V \to V$ are each diagonalizable. Is $ct$ diagonalizable for all scalars $c$? What about $t + s$? $t \circ s$?

✓ **2.17** Show that matrices of this form are not diagonalizable.

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad c \neq 0$$

**2.18** Show that each of these is diagonalizable.

**(a)** $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ $\quad x, y, z$ scalars

## II.3 Eigenvalues and Eigenvectors

In this subsection we will focus on the property of Corollary 2.4.

**3.1 Definition** A transformation $t\colon V \to V$ has a scalar *eigenvalue* $\lambda$ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

("Eigen" is German for "characteristic of" or "peculiar to"; some authors call these *characteristic* values and vectors. No authors call them "peculiar".)

**3.2 Example** The projection map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad x, y, z \in \mathbb{C}$$

has an eigenvalue of 1 associated with any eigenvector of the form

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

where $x$ and $y$ are non-0 scalars. On the other hand, 2 is not an eigenvalue of $\pi$ since no non-$\vec{0}$ vector is doubled.

That example shows why the 'non-$\vec{0}$' appears in the definition. Disallowing $\vec{0}$ as an eigenvector eliminates trivial eigenvalues.

**3.3 Example** The only transformation on the trivial space $\{\vec{0}\,\}$ is $\vec{0} \mapsto \vec{0}$. This map has no eigenvalues because there are no non-$\vec{0}$ vectors $\vec{v}$ mapped to a scalar multiple $\lambda \cdot \vec{v}$ of themselves.

**3.4 Example** Consider the homomorphism $t\colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1x \mapsto (c_0+c_1)+(c_0+c_1)x$. The range of $t$ is one-dimensional. Thus an application of $t$ to a vector in the range will simply rescale that vector: $c+cx \mapsto (2c)+(2c)x$. That is, $t$ has an eigenvalue of 2 associated with eigenvectors of the form $c+cx$ where $c \neq 0$.

This map also has an eigenvalue of 0 associated with eigenvectors of the form $c - cx$ where $c \neq 0$.

**3.5 Definition** A square matrix $T$ has a scalar *eigenvalue* $\lambda$ associated with the non-$\vec{0}$ *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

**3.6 Remark** Although this extension from maps to matrices is obvious, there is a point that must be made. Eigenvalues of a map are also the eigenvalues of matrices representing that map, and so similar matrices have the same eigenvalues. But the eigenvectors are different — similar matrices need not have the same eigenvectors.

For instance, consider again the transformation $t\colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1x \mapsto (c_0+c_1)+(c_0+c_1)x$. It has an eigenvalue of 2 associated with eigenvectors of the form $c + cx$ where $c \neq 0$. If we represent $t$ with respect to $B = \langle 1 + 1x, 1 - 1x\rangle$

$$T = \mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

then 2 is an eigenvalue of $T$, associated with these eigenvectors.

$$\{\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \mid \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2c_0 \\ 2c_1 \end{pmatrix}\} = \{\begin{pmatrix} c_0 \\ 0 \end{pmatrix} \mid c_0 \in \mathbb{C},\ c_0 \neq 0\}$$

On the other hand, representing $t$ with respect to $D = \langle 2 + 1x, 1 + 0x\rangle$ gives

$$S = \mathrm{Rep}_{D,D}(t) = \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}$$

and the eigenvectors of $S$ associated with the eigenvalue 2 are these.

$$\{\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \mid \begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2c_0 \\ 2c_1 \end{pmatrix}\} = \{\begin{pmatrix} 0 \\ c_1 \end{pmatrix} \mid c_1 \in \mathbb{C},\ c_1 \neq 0\}$$

Thus similar matrices can have different eigenvectors.

Here is an informal description of what's happening. The underlying transformation doubles the eigenvectors $\vec{v} \mapsto 2 \cdot \vec{v}$. But when the matrix representing the transformation is $T = \mathrm{Rep}_{B,B}(t)$ then it "assumes" that column vectors are representations with respect to $B$. In contrast, $S = \mathrm{Rep}_{D,D}(t)$ "assumes" that column vectors are representations with respect to $D$. So the vectors that get doubled by each matrix look different.

The next example illustrates the basic tool for finding eigenvectors and eigenvalues.

**3.7 Example** What are the eigenvalues and eigenvectors of this matrix?

$$T=\begin{pmatrix}1&2&1\\2&0&-2\\-1&2&3\end{pmatrix}$$

To find the scalars $x$ such that $T\vec{\zeta}=x\vec{\zeta}$ for non-$\vec{0}$ eigenvectors $\vec{\zeta}$, bring everything to the left-hand side

$$\begin{pmatrix}1&2&1\\2&0&-2\\-1&2&3\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}-x\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\vec{0}$$

and factor $(T-xI)\vec{\zeta}=\vec{0}$. (Note that it says $T-xI$; the expression $T-x$ doesn't make sense because $T$ is a matrix while $x$ is a scalar.) This homogeneous linear system

$$\begin{pmatrix}1-x&2&1\\2&0-x&-2\\-1&2&3-x\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a non-$\vec{0}$ solution if and only if the matrix is singular. We can determine when that happens.

$$\begin{aligned}0&=|T-xI|\\&=\begin{vmatrix}1-x&2&1\\2&0-x&-2\\-1&2&3-x\end{vmatrix}\\&=x^3-4x^2+4x\\&=x(x-2)^2\end{aligned}$$

The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$. To find the associated eigenvectors, plug in each eigenvalue. Plugging in $\lambda_1=0$ gives

$$\begin{pmatrix}1-0&2&1\\2&0-0&-2\\-1&2&3-0\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad\Longrightarrow\qquad\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}a\\-a\\a\end{pmatrix}$$

for a scalar parameter $a\neq 0$ ($a$ is non-0 because eigenvectors must be non-$\vec{0}$). In the same way, plugging in $\lambda_2=2$ gives

$$\begin{pmatrix}1-2&2&1\\2&0-2&-2\\-1&2&3-2\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\qquad\Longrightarrow\qquad\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}b\\0\\b\end{pmatrix}$$

with $b\neq 0$.

**3.8 Example** If

$$S = \begin{pmatrix} \pi & 1 \\ 0 & 3 \end{pmatrix}$$

(here $\pi$ is not a projection map, it is the number $3.14\ldots$) then

$$\left|\begin{pmatrix} \pi - x & 1 \\ 0 & 3 - x \end{pmatrix}\right| = (x - \pi)(x - 3)$$

so $S$ has eigenvalues of $\lambda_1 = \pi$ and $\lambda_2 = 3$. To find associated eigenvectors, first plug in $\lambda_1$ for $x$:

$$\begin{pmatrix} \pi - \pi & 1 \\ 0 & 3 - \pi \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

for a scalar $a \neq 0$, and then plug in $\lambda_2$:

$$\begin{pmatrix} \pi - 3 & 1 \\ 0 & 3 - 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -b/\pi - 3 \\ b \end{pmatrix}$$

where $b \neq 0$.

**3.9 Definition** The *characteristic polynomial* of a square matrix $T$ is the determinant of the matrix $T - xI$, where $x$ is a variable. The *characteristic equation* is $|T - xI| = 0$. The characteristic polynomial of a transformation $t$ is the polynomial of any $\mathrm{Rep}_{B,B}(t)$.

Exercise 30 checks that the characteristic polynomial of a transformation is well-defined, that is, any choice of basis yields the same polynomial.

**3.10 Lemma** A linear transformation on a nontrivial vector space has at least one eigenvalue.

Proof. Any root of the characteristic polynomial is an eigenvalue. Over the complex numbers, any polynomial of degree one or greater has a root. (This is the reason that in this chapter we've gone to scalars that are complex.) QED

Notice the familiar form of the sets of eigenvectors in the above examples.

**3.11 Definition** The *eigenspace* of a transformation $t$ associated with the eigenvalue $\lambda$ is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}\,) = \lambda\vec{\zeta}\,\} \cup \{\vec{0}\,\}$. The eigenspace of a matrix is defined analogously.

**3.12 Lemma** An eigenspace is a subspace.

Proof. An eigenspace must be nonempty — for one thing it contains the zero vector — and so we need only check closure. Take vectors $\vec{\zeta}_1, \ldots, \vec{\zeta}_n$ from $V_\lambda$, to show that any linear combination is in $V_\lambda$

$$\begin{aligned} t(c_1\vec{\zeta}_1 + c_2\vec{\zeta}_2 + \cdots + c_n\vec{\zeta}_n) &= c_1 t(\vec{\zeta}_1) + \cdots + c_n t(\vec{\zeta}_n) \\ &= c_1\lambda\vec{\zeta}_1 + \cdots + c_n\lambda\vec{\zeta}_n \\ &= \lambda(c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n) \end{aligned}$$

(the second equality holds even if any $\vec{\zeta}_i$ is $\vec{0}$ since $t(\vec{0}) = \lambda \cdot \vec{0} = \vec{0}$). QED

**3.13 Example** In Example 3.8 the eigenspace associated with the eigenvalue $\pi$ and the eigenspace associated with the eigenvalue 3 are these.

$$V_\pi = \{\begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{R}\} \qquad V_3 = \{\begin{pmatrix} -b/\pi - 3 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}$$

**3.14 Example** In Example 3.7, these are the eigenspaces associated with the eigenvalues 0 and 2.

$$V_0 = \{\begin{pmatrix} a \\ -a \\ a \end{pmatrix} \mid a \in \mathbb{R}\}, \qquad V_2 = \{\begin{pmatrix} b \\ 0 \\ b \end{pmatrix} \mid b \in \mathbb{R}\}.$$

**3.15 Remark** The characteristic equation is $0 = x(x-2)^2$ so in some sense 2 is an eigenvalue "twice". However there are not "twice" as many eigenvectors, in that the dimension of the eigenspace is one, not two. The next example shows a case where a number, 1, is a double root of the characteristic equation and the dimension of the associated eigenspace is two.

**3.16 Example** With respect to the standard bases, this matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

represents projection.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad x, y, z \in \mathbb{C}$$

Its eigenspace associated with the eigenvalue 0 and its eigenspace associated with the eigenvalue 1 are easy to find.

$$V_0 = \{\begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \mid c_3 \in \mathbb{C}\} \qquad V_1 = \{\begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C}\}$$

By the lemma, if two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are associated with the same eigenvalue then any linear combination of those two is also an eigenvector associated with that same eigenvalue. But, if two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are associated with different eigenvalues then the sum $\vec{v}_1 + \vec{v}_2$ need not be related to the eigenvalue of either one. In fact, just the opposite. If the eigenvalues are different then the eigenvectors are not linearly related.

**3.17 Theorem** For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

Proof. We will use induction on the number of eigenvalues. If there is no eigenvalue or only one eigenvalue then the set of associated eigenvectors is empty or is a singleton set with a non-$\vec{0}$ member, and in either case is linearly independent.

For induction, assume that the theorem is true for any set of $k$ distinct eigenvalues, suppose that $\lambda_1, \ldots, \lambda_{k+1}$ are distinct eigenvalues, and let $\vec{v}_1, \ldots, \vec{v}_{k+1}$ be associated eigenvectors. If $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} = \vec{0}$ then after multiplying both sides of the displayed equation by $\lambda_{k+1}$, applying the map or matrix to both sides of the displayed equation, and subtracting the first result from the second, we have this.

$$c_1(\lambda_{k+1} - \lambda_1)\vec{v}_1 + \cdots + c_k(\lambda_{k+1} - \lambda_k)\vec{v}_k + c_{k+1}(\lambda_{k+1} - \lambda_{k+1})\vec{v}_{k+1} = \vec{0}$$

The induction hypothesis now applies: $c_1(\lambda_{k+1} - \lambda_1) = 0, \ldots, c_k(\lambda_{k+1} - \lambda_k) = 0$. Thus, as all the eigenvalues are distinct, $c_1, \ldots, c_k$ are all 0. Finally, now $c_{k+1}$ must be 0 because we are left with the equation $\vec{v}_{k+1} \neq \vec{0}$. QED

**3.18 Example** The eigenvalues of

$$\begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{pmatrix}$$

are distinct: $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. A set of associated eigenvectors like

$$\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\}$$

is linearly independent.

**3.19 Corollary** An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Proof. Form a basis of eigenvectors. Apply Corollary 2.4. QED

## Exercises

**3.20** For each, find the characteristic polynomial and the eigenvalues.

**(a)** $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 0 & 3 \\ 7 & 0 \end{pmatrix}$ **(d)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**(e)** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

✓ **3.21** For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

**(a)** $\begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

**3.22** Find the characteristic equation, and the eigenvalues and associated eigenvectors for this matrix. *Hint.* The eigenvalues are complex.

$$\begin{pmatrix} -2 & -1 \\ 5 & 2 \end{pmatrix}$$

**3.23** Find the characteristic polynomial, the eigenvalues, and the associated eigenvectors of this matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

✓ **3.24** For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

**(a)** $\begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$

✓ **3.25** Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be

$$a_0 + a_1x + a_2x^2 \mapsto (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find its eigenvalues and the associated eigenvectors.

**3.26** Find the eigenvalues and eigenvectors of this map $t\colon \mathcal{M}_2 \to \mathcal{M}_2$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 2c & a+c \\ b-2c & d \end{pmatrix}$$

✓ **3.27** Find the eigenvalues and associated eigenvectors of the differentiation operator $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$.

**3.28** Prove that the eigenvalues of a triangular matrix (upper or lower triangular) are the entries on the diagonal.

✓ **3.29** Find the formula for the characteristic polynomial of a $2\times 2$ matrix.

**3.30** Prove that the characteristic polynomial of a transformation is well-defined.

✓ **3.31** **(a)** Can any non-$\vec{0}$ vector in any nontrivial vector space be a eigenvector? That is, given a $\vec{v} \neq \vec{0}$ from a nontrivial $V$, is there a transformation $t\colon V \to V$ and a scalar $\lambda \in \mathbb{R}$ such that $t(\vec{v}) = \lambda\vec{v}$?

**(b)** Given a scalar $\lambda$, can any non-$\vec{0}$ vector in any nontrivial vector space be an eigenvector associated with the eigenvalue $\lambda$?

✓ **3.32** Suppose that $t\colon V \to V$ and $T = \text{Rep}_{B,B}(t)$. Prove that the eigenvectors of $T$ associated with $\lambda$ are the non-$\vec{0}$ vectors in the kernel of the map represented (with respect to the same bases) by $T - \lambda I$.

**3.33** Prove that if $a, \ldots, d$ are all integers and $a + b = c + d$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has integral eigenvalues, namely $a + b$ and $a - c$.

✓ **3.34** Prove that if $T$ is nonsingular and has eigenvalues $\lambda_1, \ldots, \lambda_n$ then $T^{-1}$ has eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Is the converse true?

✓ **3.35** Suppose that $T$ is $n\times n$ and $c, d$ are scalars.

**(a)** Prove that if $T$ has the eigenvalue $\lambda$ with an associated eigenvector $\vec{v}$ then $\vec{v}$ is an eigenvector of $cT + dI$ associated with eigenvalue $c\lambda + d$.

**(b)** Prove that if $T$ is diagonalizable then so is $cT + dI$.

✓ **3.36** Show that $\lambda$ is an eigenvalue of $T$ if and only if the map represented by $T - \lambda I$ is not an isomorphism.

**3.37** [Strang 80]

**(a)** Show that if $\lambda$ is an eigenvalue of $A$ then $\lambda^k$ is an eigenvalue of $A^k$.

**(b)** What is wrong with this proof generalizing that? "If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue for $B$, then $\lambda\mu$ is an eigenvalue for $AB$, for, if $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$ then $AB\vec{x} = A\mu\vec{x} = \mu A\vec{x}\mu\lambda\vec{x}$"?

**3.38** Do matrix-equivalent matrices have the same eigenvalues?

**3.39** Show that a square matrix with real entries and an odd number of rows has at least one real eigenvalue.

**3.40** Diagonalize.

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$$

**3.41** Suppose that $P$ is a nonsingular $n \times n$ matrix. Show that the *similarity transformation* map $t_P \colon \mathcal{M}_{n \times n} \to \mathcal{M}_{n \times n}$ sending $T \mapsto PTP^{-1}$ is an isomorphism.

**? 3.42** Show that if $A$ is an $n$ square matrix and each row (column) sums to $c$ then $c$ is a characteristic root of $A$. [Math. Mag., Nov. 1967]

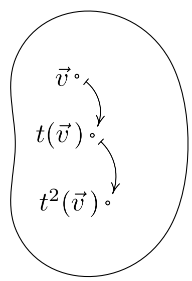
# III Nilpotence

The goal of this chapter is to show that every square matrix is similar to one that is a sum of two kinds of simple matrices. The prior section focused on the first kind, diagonal matrices. We now consider the other kind.

## III.1 Self-Composition

*This subsection is optional, although it is necessary for later material in this section and in the next one.*

A linear transformations $t\colon V \to V$, because it has the same domain and codomain, can be iterated.* That is, compositions of $t$ with itself such as $t^2 = t \circ t$ and $t^3 = t \circ t \circ t$ are defined.




Note that this power notation for the linear transformation functions dovetails with the notation that we've used earlier for their square matrix representations because if $\mathrm{Rep}_{B,B}(t) = T$ then $\mathrm{Rep}_{B,B}(t^j) = T^j$.

**1.1 Example** For the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$a + bx + cx^2 + dx^3 \overset{d/dx}{\longmapsto} b + 2cx + 3dx^2$$

the second power is the second derivative

$$a + bx + cx^2 + dx^3 \overset{d^2/dx^2}{\longmapsto} 2c + 6dx$$

the third power is the third derivative

$$a + bx + cx^2 + dx^3 \overset{d^3/dx^3}{\longmapsto} 6d$$

and any higher power is the zero map.

**1.2 Example** This transformation of the space of $2\times 2$ matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{t}{\longmapsto} \begin{pmatrix} b & a \\ d & 0 \end{pmatrix}$$

* More information on function interation is in the appendix.

has this second power

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{t^2}{\longmapsto} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

and this third power.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{t^3}{\longmapsto} \begin{pmatrix} b & a \\ 0 & 0 \end{pmatrix}$$

After that, $t^4 = t^2$ and $t^5 = t^3$, etc.

These examples suggest that on iteration more and more zeros appear until there is a settling down. The next result makes this precise.

**1.3 Lemma** For any transformation $t\colon V \to V$, the rangespaces of the powers form a descending chain

$$V \supseteq \mathscr{R}(t) \supseteq \mathscr{R}(t^2) \supseteq \cdots$$

and the nullspaces form an ascending chain.

$$\{\vec{0}\,\} \subseteq \mathscr{N}(t) \subseteq \mathscr{N}(t^2) \subseteq \cdots$$

Further, there is a $k$ such that for powers less than $k$ the subsets are proper (if $j < k$ then $\mathscr{R}(t^j) \supset \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) \subset \mathscr{N}(t^{j+1})$), while for powers greater than $k$ the sets are equal (if $j \geq k$ then $\mathscr{R}(t^j) = \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) = \mathscr{N}(t^{j+1})$).

Proof. We will do the rangespace half and leave the rest for Exercise 13. Recall, however, that for any map the dimension of its rangespace plus the dimension of its nullspace equals the dimension of its domain. So if the rangespaces shrink then the nullspaces must grow.

That the rangespaces form chains is clear because if $\vec{w} \in \mathscr{R}(t^{j+1})$, so that $\vec{w} = t^{j+1}(\vec{v})$, then $\vec{w} = t^j(\,t(\vec{v})\,)$ and so $\vec{w} \in \mathscr{R}(t^j)$. To verify the "further" property, first observe that if any pair of rangespaces in the chain are equal $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1})$ then all subsequent ones are also equal $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$, etc. This is because if $t\colon \mathscr{R}(t^{k+1}) \to \mathscr{R}(t^{k+2})$ is the same map, with the same domain, as $t\colon \mathscr{R}(t^k) \to \mathscr{R}(t^{k+1})$ and it therefore has the same range: $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$ (and induction shows that it holds for all higher powers). So if the chain of rangespaces ever stops being strictly decreasing then it is stable from that point onward.

But the chain must stop decreasing. Each rangespace is a subspace of the one before it. For it to be a proper subspace it must be of strictly lower dimension (see Exercise 11). These spaces are finite-dimensional and so the chain can fall for only finitely-many steps, that is, the power $k$ is at most the dimension of $V$. QED

**1.4 Example** The derivative map $a + bx + cx^2 + dx^3 \overset{d/dx}{\longmapsto} b + 2cx + 3dx^2$ of Example 1.1 has this chain of rangespaces

$$\mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1 \supset \mathcal{P}_0 \supset \{\vec{0}\,\} = \{\vec{0}\,\} = \cdots$$

and this chain of nullspaces.

$$\{\vec{0}\,\} \subset \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 = \mathcal{P}_3 = \cdots$$

**1.5 Example** The transformation $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

has $\mathbb{C}^3 \supset \mathscr{R}(\pi) = \mathscr{R}(\pi^2) = \cdots$ and $\{\vec{0}\,\} \subset \mathscr{N}(\pi) = \mathscr{N}(\pi^2) = \cdots$.

**1.6 Example** Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the map $c_0 + c_1x + c_2x^2 \mapsto 2c_0 + c_2x$. As the lemma describes, on iteration the rangespace shrinks

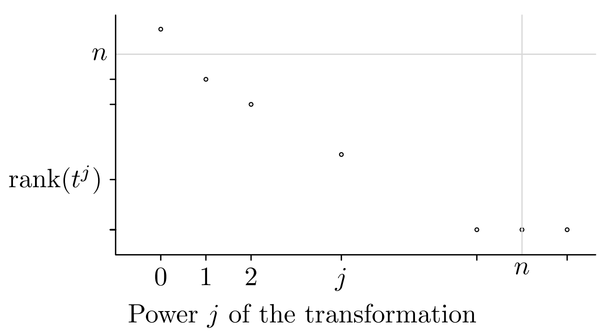
$$\mathscr{R}(t^0) = \mathcal{P}_2 \quad \mathscr{R}(t) = \{a + bx \mid a, b \in \mathbb{C}\} \quad \mathscr{R}(t^2) = \{a \mid a \in \mathbb{C}\}$$

and then stabilizes $\mathscr{R}(t^2) = \mathscr{R}(t^3) = \cdots$, while the nullspace grows

$$\mathscr{N}(t^0) = \{0\} \quad \mathscr{N}(t) = \{cx \mid c \in \mathbb{C}\} \quad \mathscr{N}(t^2) = \{cx + d \mid c, d \in \mathbb{C}\}$$

and then stabilizes $\mathscr{N}(t^2) = \mathscr{N}(t^3) = \cdots$.

This graph illustrates Lemma 1.3. The horizontal axis gives the power $j$ of a transformation. The vertical axis gives the dimension of the rangespace of $t^j$ as the distance above zero — and thus also shows the dimension of the nullspace as the distance below the gray horizontal line, because the two add to the dimension $n$ of the domain.




As sketched, on iteration the rank falls and with it the nullity grows until the two reach a steady state. This state must be reached by the $n$-th iterate. The steady state's distance above zero is the dimension of the generalized rangespace and its distance below $n$ is the dimension of the generalized nullspace.

**1.7 Definition** Let $t$ be a transformation on an $n$-dimensional space. The *generalized rangespace* (or the *closure of the rangespace*) is $\mathscr{R}_\infty(t) = \mathscr{R}(t^n)$ The *generalized nullspace* (or the *closure of the nullspace*) is $\mathscr{N}_\infty(t) = \mathscr{N}(t^n)$.

**Exercises**

**1.8** Give the chains of rangespaces and nullspaces for the zero and identity transformations.

**1.9** For each map, give the chain of rangespaces and the chain of nullspaces, and the generalized rangespace and the generalized nullspace.
(a) $t_0 \colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx^2$
(b) $t_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ a \end{pmatrix}$$

(c) $t_2 \colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx + ax^2$
(d) $t_3 \colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ a \\ b \end{pmatrix}$$

**1.10** Prove that function composition is associative $(t \circ t) \circ t = t \circ (t \circ t)$ and so we can write $t^3$ without specifying a grouping.

**1.11** Check that a subspace must be of dimension less than or equal to the dimension of its superspace. Check that if the subspace is proper (the subspace does not equal the superspace) then the dimension is strictly less. *(This is used in the proof of Lemma 1.3.)*

**1.12** Prove that the generalized rangespace $\mathscr{R}_\infty(t)$ is the entire space, and the generalized nullspace $\mathscr{N}_\infty(t)$ is trivial, if the transformation $t$ is nonsingular. Is this 'only if' also?

**1.13** Verify the nullspace half of Lemma 1.3.

**1.14** Give an example of a transformation on a three dimensional space whose range has dimension two. What is its nullspace? Iterate your example until the rangespace and nullspace stabilize.

**1.15** Show that the rangespace and nullspace of a linear transformation need not be disjoint. Are they ever disjoint?

## III.2 Strings

*This subsection is optional, and requires material from the optional Direct Sum subsection.*

The prior subsection shows that as $j$ increases, the dimensions of the $\mathscr{R}(t^j)$'s fall while the dimensions of the $\mathscr{N}(t^j)$'s rise, in such a way that this rank and nullity split the dimension of $V$. Can we say more; do the two split a basis — is $V = \mathscr{R}(t^j) \oplus \mathscr{N}(t^j)$?

The answer is yes for the smallest power $j = 0$ since $V = \mathscr{R}(t^0) \oplus \mathscr{N}(t^0) = V \oplus \{\vec{0}\}$. The answer is also yes at the other extreme.

**2.1 Lemma** Where $t \colon V \to V$ is a linear transformation, the space is the direct sum $V = \mathscr{R}_\infty(t) \oplus \mathscr{N}_\infty(t)$. That is, both $\dim(V) = \dim(\mathscr{R}_\infty(t)) + \dim(\mathscr{N}_\infty(t))$ and $\mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t) = \{\vec{0}\}$.

Proof. We will verify the second sentence, which is equivalent to the first. The first clause, that the dimension $n$ of the domain of $t^n$ equals the rank of $t^n$ plus the nullity of $t^n$, holds for any transformation and so we need only verify the second clause.

Assume that $\vec{v} \in \mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t) = \mathscr{R}(t^n) \cap \mathscr{N}(t^n)$, to prove that $\vec{v}$ is $\vec{0}$. Because $\vec{v}$ is in the nullspace, $t^n(\vec{v}) = \vec{0}$. On the other hand, because $\mathscr{R}(t^n) = \mathscr{R}(t^{n+1})$, the map $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is a dimension-preserving homomorphism and therefore is one-to-one. A composition of one-to-one maps is one-to-one, and so $t^n : \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one. But now — because only $\vec{0}$ is sent by a one-to-one linear map to $\vec{0}$ — the fact that $t^n(\vec{v}) = \vec{0}$ implies that $\vec{v} = \vec{0}$. QED

**2.2 Note** Technically we should distinguish the map $t\colon V \to V$ from the map $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ because the domains or codomains might differ. The second one is said to be the *restriction*[*] of $t$ to $\mathscr{R}(t^k)$. We shall use later a point from that proof about the restriction map, namely that it is nonsingular.

In contrast to the $j = 0$ and $j = n$ cases, for intermediate powers the space $V$ might not be the direct sum of $\mathscr{R}(t^j)$ and $\mathscr{N}(t^j)$. The next example shows that the two can have a nontrivial intersection.

**2.3 Example** Consider the transformation of $\mathbb{C}^2$ defined by this action on the elements of the standard basis.

$$\begin{pmatrix}1\\0\end{pmatrix} \stackrel{n}{\longmapsto} \begin{pmatrix}0\\1\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix} \stackrel{n}{\longmapsto} \begin{pmatrix}0\\0\end{pmatrix} \qquad N = \text{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(n) = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

The vector

$$\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

is in both the rangespace and nullspace. Another way to depict this map's action is with a *string*.

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{0}$$

**2.4 Example** A map $\hat{n}\colon \mathbb{C}^4 \to \mathbb{C}^4$ whose action on $\mathcal{E}_4$ is given by the string

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{e}_3 \mapsto \vec{e}_4 \mapsto \vec{0}$$

has $\mathscr{R}(\hat{n}) \cap \mathscr{N}(\hat{n})$ equal to the span $[\{\vec{e}_4\}]$, has $\mathscr{R}(\hat{n}^2) \cap \mathscr{N}(\hat{n}^2) = [\{\vec{e}_3, \vec{e}_4\}]$, and has $\mathscr{R}(\hat{n}^3) \cap \mathscr{N}(\hat{n}^3) = [\{\vec{e}_4\}]$. The matrix representation is all zeros except for some subdiagonal ones.

$$\hat{N} = \text{Rep}_{\mathcal{E}_4,\mathcal{E}_4}(\hat{n}) = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

[*] More information on map restrictions is in the appendix.

**2.5 Example** Transformations can act via more than one string. A transformation $t$ acting on a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_5 \rangle$ by

$$\begin{array}{ccccccc} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{\beta}_3 & \mapsto & \vec{0} \\ \vec{\beta}_4 & \mapsto & \vec{\beta}_5 & \mapsto & \vec{0} & & \end{array}$$

is represented by a matrix that is all zeros except for blocks of subdiagonal ones

$$\text{Rep}_{B,B}(t) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

(the lines just visually organize the blocks).

In those three examples all vectors are eventually transformed to zero.

**2.6 Definition** A *nilpotent* transformation is one with a power that is the zero map. A *nilpotent matrix* is one with a power that is the zero matrix. In either case, the least such power is the *index of nilpotency*.

**2.7 Example** In Example 2.3 the index of nilpotency is two. In Example 2.4 it is four. In Example 2.5 it is three.

**2.8 Example** The differentiation map $d/dx\colon \mathcal{P}_2 \to \mathcal{P}_2$ is nilpotent of index three since the third derivative of any quadratic polynomial is zero. This map's action is described by the string $x^2 \mapsto 2x \mapsto 2 \mapsto 0$ and taking the basis $B = \langle x^2, 2x, 2 \rangle$ gives this representation.

$$\text{Rep}_{B,B}(d/dx) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Not all nilpotent matrices are all zeros except for blocks of subdiagonal ones.

**2.9 Example** With the matrix $\hat{N}$ from Example 2.4, and this four-vector basis

$$D = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

a change of basis operation produces this representation with respect to $D, D$.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -3 & -2 & 5 & 0 \\ -2 & -1 & 3 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}$$

The new matrix is nilpotent; it's fourth power is the zero matrix since

$$(P\hat{N}P^{-1})^4 = P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} = P\hat{N}^4P^{-1}$$

and $\hat{N}^4$ is the zero matrix.

The goal of this subsection is Theorem 2.13, which shows that the prior example is prototypical in that every nilpotent matrix is similar to one that is all zeros except for blocks of subdiagonal ones.

**2.10 Definition** Let $t$ be a nilpotent transformation on $V$. A *$t$-string generated by $\vec{v} \in V$* is a sequence $\langle \vec{v}, t(\vec{v}), \ldots, t^{k-1}(\vec{v})\rangle$. This sequence has *length $k$*. A *$t$-string basis* is a basis that is a concatenation of $t$-strings.

**2.11 Example** In Example 2.5, the $t$-strings $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3\rangle$ and $\langle \vec{\beta}_4, \vec{\beta}_5\rangle$, of length three and two, can be concatenated to make a basis for the domain of $t$.

**2.12 Lemma** If a space has a $t$-string basis then the longest string in it has length equal to the index of nilpotency of $t$.

Proof. Suppose not. Those strings cannot be longer; if the index is $k$ then $t^k$ sends any vector — including those starting the string — to $\vec{0}$. So suppose instead that there is a transformation $t$ of index $k$ on some space, such that the space has a $t$-string basis where all of the strings are shorter than length $k$. Because $t$ has index $k$, there is a vector $\vec{v}$ such that $t^{k-1}(\vec{v}) \neq \vec{0}$. Represent $\vec{v}$ as a linear combination of basis elements and apply $t^{k-1}$. We are supposing that $t^{k-1}$ sends each basis element to $\vec{0}$ but that it does not send $\vec{v}$ to $\vec{0}$. That is impossible. QED

We shall show that every nilpotent map has an associated string basis. Then our goal theorem, that every nilpotent matrix is similar to one that is all zeros except for blocks of subdiagonal ones, is immediate, as in Example 2.5.

Looking for a counterexample, a nilpotent map without an associated string basis that is disjoint, will suggest the idea for the proof. Consider the map $t\colon \mathbb{C}^5 \to \mathbb{C}^5$ with this action.

$$\begin{array}{c} \vec{e}_1 \searrow \\ \qquad\vec{e}_3 \mapsto \vec{0} \\ \vec{e}_2 \nearrow \\ \\ \vec{e}_4 \mapsto \vec{e}_5 \mapsto \vec{0} \end{array} \qquad \mathrm{Rep}_{\mathcal{E}_5,\mathcal{E}_5}(t) = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \end{pmatrix}$$

Even after ommitting the zero vector, these three strings aren't disjoint, but that doesn't end hope of finding a $t$-string basis. It only means that $\mathcal{E}_5$ will not do for the string basis.

To find a basis that will do, we first find the number and lengths of its strings. Since $t$'s index of nilpotency is two, Lemma 2.12 says that at least one string in the basis has length two. Thus the map must act on a string basis in one of these two ways.

$$\begin{array}{lll} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \\ \vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0} \\ \vec{\beta}_5 \mapsto \vec{0} \end{array} \qquad \begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \\ \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_4 \mapsto \vec{0} \\ \vec{\beta}_5 \mapsto \vec{0} \end{array}$$

Now, the key point. A transformation with the left-hand action has a nullspace of dimension three since that's how many basis vectors are sent to zero. A transformation with the right-hand action has a nullspace of dimension four. Using the matrix representation above, calculation of $t$'s nullspace

$$\mathscr{N}(t) = \{ \begin{pmatrix} x \\ -x \\ z \\ 0 \\ r \end{pmatrix} \bigm| x, z, r \in \mathbb{C} \}$$

shows that it is three-dimensional, meaning that we want the left-hand action.

To produce a string basis, first pick $\vec{\beta}_2$ and $\vec{\beta}_4$ from $\mathscr{R}(t) \cap \mathscr{N}(t)$

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(other choices are possible, just be sure that $\{\vec{\beta}_2, \vec{\beta}_4\}$ is linearly independent). For $\vec{\beta}_5$ pick a vector from $\mathscr{N}(t)$ that is not in the span of $\{\vec{\beta}_2, \vec{\beta}_4\}$.

$$\vec{\beta}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Finally, take $\vec{\beta}_1$ and $\vec{\beta}_3$ such that $t(\vec{\beta}_1) = \vec{\beta}_2$ and $t(\vec{\beta}_3) = \vec{\beta}_4$.

$$\vec{\beta}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Now, with respect to $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_5 \rangle$, the matrix of $t$ is as desired.

$$\mathrm{Rep}_{B,B}(t) = \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

**2.13 Theorem** Any nilpotent transformation $t$ is associated with a $t$-string basis. While the basis is not unique, the number and the length of the strings is determined by $t$.

This illustrates the proof. Basis vectors are categorized into kind 1, kind 2, and kind 3. They are also shown as squares or circles, according to whether they are in the nullspace or not.

$$\begin{array}{lllllllllll}
3 & \mapsto & 1 & \mapsto & \cdots & & & & \cdots \mapsto & 1 & \mapsto \boxed{1} \mapsto \vec{0} \\
3 & \mapsto & 1 & \mapsto & \cdots & & & & \cdots \mapsto & 1 & \mapsto \boxed{1} \mapsto \vec{0} \\
 & \vdots & & & & & & & & & \\
3 & \mapsto & 1 & \mapsto & \cdots & \mapsto & 1 & \mapsto & \boxed{1} & \mapsto & \vec{0} \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & & \\
 & \vdots & & & & & & & & & \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & &
\end{array}$$

Proof. Fix a vector space $V$; we will argue by induction on the index of nilpotency of $t\colon V \to V$. If that index is 1 then $t$ is the zero map and any basis is a string basis $\vec{\beta}_1 \mapsto \vec{0}, \ldots, \vec{\beta}_n \mapsto \vec{0}$. For the inductive step, assume that the theorem holds for any transformation with an index of nilpotency between 1 and $k-1$ and consider the index $k$ case.

First observe that the restriction to the rangespace $t\colon \mathscr{R}(t) \to \mathscr{R}(t)$ is also nilpotent, of index $k-1$. Apply the inductive hypothesis to get a string basis for $\mathscr{R}(t)$, where the number and length of the strings is determined by $t$.

$$B = \langle \vec{\beta}_1, t(\vec{\beta}_1), \ldots, t^{h_1}(\vec{\beta}_1) \rangle^\frown \langle \vec{\beta}_2, \ldots, t^{h_2}(\vec{\beta}_2) \rangle^\frown \cdots^\frown \langle \vec{\beta}_i, \ldots, t^{h_i}(\vec{\beta}_i) \rangle$$

(In the illustration these are the basis vectors of kind 1, so there are $i$ strings shown with this kind of basis vector.)

Second, note that taking the final nonzero vector in each string gives a basis $C = \langle t^{h_1}(\vec{\beta}_1), \ldots, t^{h_i}(\vec{\beta}_i) \rangle$ for $\mathscr{R}(t) \cap \mathscr{N}(t)$. (These are illustrated with 1's in squares.) For, a member of $\mathscr{R}(t)$ is mapped to zero if and only if it is a linear combination of those basis vectors that are mapped to zero. Extend $C$ to a basis for all of $\mathscr{N}(t)$.

$$\hat{C} = C^\frown \langle \vec{\xi}_1, \ldots, \vec{\xi}_p \rangle$$

(The $\vec{\xi}$'s are the vectors of kind 2 so that $\hat{C}$ is the set of squares.) While many choices are possible for the $\vec{\xi}$'s, their number $p$ is determined by the map $t$ as it is the dimension of $\mathscr{N}(t)$ minus the dimension of $\mathscr{R}(t) \cap \mathscr{N}(t)$.

Finally, $B^\frown\hat{C}$ is a basis for $\mathscr{R}(t)+\mathscr{N}(t)$ because any sum of something in the rangespace with something in the nullspace can be represented using elements of $B$ for the rangespace part and elements of $\hat{C}$ for the part from the nullspace. Note that

$$\begin{aligned}\dim\bigl(\mathscr{R}(t)+\mathscr{N}(t)\bigr) &= \dim(\mathscr{R}(t))+\dim(\mathscr{N}(t))-\dim(\mathscr{R}(t)\cap\mathscr{N}(t))\\ &= \operatorname{rank}(t)+\operatorname{nullity}(t)-i\\ &= \dim(V)-i\end{aligned}$$

and so $B^\frown\hat{C}$ can be extended to a basis for all of $V$ by the addition of $i$ more vectors. Specifically, remember that each of $\vec{\beta}_1,\ldots,\vec{\beta}_i$ is in $\mathscr{R}(t)$, and extend $B^\frown\hat{C}$ with vectors $\vec{v}_1,\ldots,\vec{v}_i$ such that $t(\vec{v}_1)=\vec{\beta}_1,\ldots,t(\vec{v}_i)=\vec{\beta}_i$. (In the illustration, these are the 3's.) The check that linear independence is preserved by this extension is Exercise 29. QED

**2.14 Corollary** Every nilpotent matrix is similar to a matrix that is all zeros except for blocks of subdiagonal ones. That is, every nilpotent map is represented with respect to some basis by such a matrix.

This form is unique in the sense that if a nilpotent matrix is similar to two such matrices then those two simply have their blocks ordered differently. Thus this is a canonical form for the similarity classes of nilpotent matrices provided that we order the blocks, say, from longest to shortest.

**2.15 Example** The matrix

$$M=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$$

has an index of nilpotency of two, as this calculation shows.

| $p$ | $M^p$ | $\mathscr{N}(M^p)$ |
|---|---|---|
| 1 | $M=\begin{pmatrix}1&-1\\1&-1\end{pmatrix}$ | $\{\begin{pmatrix}x\\x\end{pmatrix}\mid x\in\mathbb{C}\}$ |
| 2 | $M^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$ | $\mathbb{C}^2$ |

The calculation also describes how a map $m$ represented by $M$ must act on any string basis. With one map application the nullspace has dimension one and so one vector of the basis is sent to zero. On a second application, the nullspace has dimension two and so the other basis vector is sent to zero. Thus, the action of the map is $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$ and the canonical form of the matrix is this.

$$\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

We can exhibit such a $m$-string basis and the change of basis matrices witnessing the matrix similarity. For the basis, take $M$ to represent $m$ with respect to the standard bases, pick a $\vec{\beta}_2 \in \mathscr{N}(m)$ and also pick a $\vec{\beta}_1$ so that $m(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(If we take $M$ to be a representative with respect to some nonstandard bases then this picking step is just more messy.) Recall the similarity diagram.

$$\begin{array}{ccc} \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[M]{m} & \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow P & & \text{id}\big\downarrow P \\ C^2_{\text{w.r.t. } B} & \xrightarrow{m} & \mathbb{C}^2_{\text{w.r.t. } B} \end{array}$$

The canonical form equals $\text{Rep}_{B,B}(m) = PMP^{-1}$, where

$$P^{-1} = \text{Rep}_{B,\mathcal{E}_2}(\text{id}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad P = (P^{-1})^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and the verification of the matrix calculation is routine.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

**2.16 Example** The matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \end{pmatrix}$$

is nilpotent. These calculations show the nullspaces growing.

| $p$ | $N^p$ | $\mathscr{N}(N^p)$ |
|---|---|---|
| 1 | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & -1 \end{pmatrix}$ | $\{\begin{pmatrix} 0 \\ 0 \\ u-v \\ u \\ v \end{pmatrix} \mid u, v \in \mathbb{C}\}$ |
| 2 | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | $\{\begin{pmatrix} 0 \\ y \\ z \\ u \\ v \end{pmatrix} \mid y, z, u, v \in \mathbb{C}\}$ |
| 3 | *–zero matrix–* | $\mathbb{C}^5$ |

That table shows that any string basis must satisfy: the nullspace after one map application has dimension two so two basis vectors are sent directly to zero, the nullspace after the second application has dimension four so two additional basis vectors are sent to zero by the second iteration, and the nullspace after three applications is of dimension five so the final basis vector is sent to zero in three hops.

$$\begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_4 \mapsto \vec{\beta}_5 \mapsto \vec{0} \end{array}$$

To produce such a basis, first pick two independent vectors from $\mathscr{N}(n)$

$$\vec{\beta}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{\beta}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

then add $\vec{\beta}_2, \vec{\beta}_4 \in \mathscr{N}(n^2)$ such that $n(\vec{\beta}_2) = \vec{\beta}_3$ and $n(\vec{\beta}_4) = \vec{\beta}_5$

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and finish by adding $\vec{\beta}_1 \in \mathscr{N}(n^3) = \mathbb{C}^5)$ such that $n(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

## Exercises

✓ **2.17** What is the index of nilpotency of the *left-shift* operator, here acting on the space of triples of reals?

$$(x, y, z) \mapsto (0, x, y)$$

✓ **2.18** For each string basis state the index of nilpotency and give the dimension of the rangespace and nullspace of each iteration of the nilpotent map.

**(a)** $\begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0} \\ \vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0} \end{array}$

**(b)** $\begin{array}{l} \vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0} \\ \vec{\beta}_4 \mapsto \vec{0} \\ \vec{\beta}_5 \mapsto \vec{0} \\ \vec{\beta}_6 \mapsto \vec{0} \end{array}$

**(c)** $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$

Also give the canonical form of the matrix.

**2.19** Decide which of these matrices are nilpotent.

**(a)** $\begin{pmatrix}-2&4\\-1&2\end{pmatrix}$ **(b)** $\begin{pmatrix}3&1\\1&3\end{pmatrix}$ **(c)** $\begin{pmatrix}-3&2&1\\-3&2&1\\-3&2&1\end{pmatrix}$ **(d)** $\begin{pmatrix}1&1&4\\3&0&-1\\5&2&7\end{pmatrix}$

**(e)** $\begin{pmatrix}45&-22&-19\\33&-16&-14\\69&-34&-29\end{pmatrix}$

✓ **2.20** Find the canonical form of this matrix.

$$\begin{pmatrix}0&1&1&0&1\\0&0&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$

✓ **2.21** Consider the matrix from Example 2.16.

**(a)** Use the action of the map on the string basis to give the canonical form.

**(b)** Find the change of basis matrices that bring the matrix to canonical form.

**(c)** Use the answer in the prior item to check the answer in the first item.

✓ **2.22** Each of these matrices is nilpotent.

**(a)** $\begin{pmatrix}1/2&-1/2\\1/2&-1/2\end{pmatrix}$ **(b)** $\begin{pmatrix}0&0&0\\0&-1&1\\0&-1&1\end{pmatrix}$ **(c)** $\begin{pmatrix}-1&1&-1\\1&0&1\\1&-1&1\end{pmatrix}$

Put each in canonical form.

**2.23** Describe the effect of left or right multiplication by a matrix that is in the canonical form for nilpotent matrices.

**2.24** Is nilpotence invariant under similarity? That is, must a matrix similar to a nilpotent matrix also be nilpotent? If so, with the same index?

✓ **2.25** Show that the only eigenvalue of a nilpotent matrix is zero.

**2.26** Is there a nilpotent transformation of index three on a two-dimensional space?

**2.27** In the proof of Theorem 2.13, why isn't the proof's base case that the index of nilpotency is zero?

✓ **2.28** Let $t\colon V\to V$ be a linear transformation and suppose $\vec{v}\in V$ is such that $t^k(\vec{v})=\vec{0}$ but $t^{k-1}(\vec{v})\neq\vec{0}$. Consider the $t$-string $\langle\vec{v},t(\vec{v}),\dots,t^{k-1}(\vec{v})\rangle$.

**(a)** Prove that $t$ is a transformation on the span of the set of vectors in the string, that is, prove that $t$ restricted to the span has a range that is a subset of the span. We say that the span is a *$t$-invariant* subspace.

**(b)** Prove that the restriction is nilpotent.

**(c)** Prove that the $t$-string is linearly independent and so is a basis for its span.

**(d)** Represent the restriction map with respect to the $t$-string basis.

**2.29** Finish the proof of Theorem 2.13.

**2.30** Show that the terms 'nilpotent transformation' and 'nilpotent matrix', as given in Definition 2.6, fit with each other: a map is nilpotent if and only if it is represented by a nilpotent matrix. (Is it that a transformation is nilpotent if an only if there is a basis such that the map's representation with respect to that basis is a nilpotent matrix, or that any representation is a nilpotent matrix?)

**2.31** Let $T$ be nilpotent of index four. How big can the rangespace of $T^3$ be?

**2.32** Recall that similar matrices have the same eigenvalues. Show that the converse does not hold.

**2.33** Prove a nilpotent matrix is similar to one that is all zeros except for blocks of super-diagonal ones.

✓ **2.34** Prove that if a transformation has the same rangespace as nullspace. then the dimension of its domain is even.

**2.35** Prove that if two nilpotent matrices commute then their product and sum are also nilpotent.

**2.36** Consider the transformation of $\mathcal{M}_{n\times n}$ given by $t_S(T) = ST - TS$ where $S$ is an $n\times n$ matrix. Prove that if $S$ is nilpotent then so is $t_S$.

**2.37** Show that if $N$ is nilpotent then $I - N$ is invertible. Is that 'only if' also?

# IV Jordan Form

*This section uses material from three optional subsections: Direct Sum, Determinants Exist, and Other Formulas for the Determinant.*

The chapter on linear maps shows that every $h\colon V \to W$ can be represented by a partial-identity matrix with respect to some bases $B \subset V$ and $D \subset W$. This chapter revisits this issue in the special case that the map is a linear transformation $t\colon V \to V$. Of course, the general result still applies but with the codomain and domain equal we naturally ask about having the two bases also be equal. That is, we want a canonical form to represent transformations as $\mathrm{Rep}_{B,B}(t)$.

After a brief review section, we began by noting that a block partial identity form matrix is not always obtainable in this $B, B$ case. We therefore considered the natural generalization, diagonal matrices, and showed that if its eigenvalues are distinct then a map or matrix can be diagonalized. But we also gave an example of a matrix that cannot be diagonalized and in the section prior to this one we developed that example. We showed that a linear map is nilpotent — if we take higher and higher powers of the map or matrix then we eventually get the zero map or matrix — if and only if there is a basis on which it acts via disjoint strings. That led to a canonical form for nilpotent matrices.

Now, this section concludes the chapter. We will show that the two cases we've studied are exhaustive in that for any linear transformation there is a basis such that the matrix representation $\mathrm{Rep}_{B,B}(t)$ is the sum of a diagonal matrix and a nilpotent matrix in its canonical form.

## IV.1 Polynomials of Maps and Matrices

Recall that the set of square matrices is a vector space under entry-by-entry addition and scalar multiplication and that this space $\mathcal{M}_{n\times n}$ has dimension $n^2$. Thus, for any $n\times n$ matrix $T$ the $n^2+1$-member set $\{I, T, T^2, \dots, T^{n^2}\}$ is linearly dependent and so there are scalars $c_0, \dots, c_{n^2}$ such that $c_{n^2}T^{n^2} + \cdots + c_1T + c_0I$ is the zero matrix.

**1.1 Remark** This observation is small but important. It says that every transformation exhibits a generalized nilpotency: the powers of a square matrix cannot climb forever without a "repeat".

**1.2 Example** Rotation of plane vectors $\pi/6$ radians counterclockwise is represented with respect to the standard basis by

$$T = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

and verifying that $0T^4 + 0T^3 + 1T^2 - 2T - 1I$ equals the zero matrix is easy.

**1.3 Definition** For any polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$, where $t$ is a linear transformation then $f(t)$ is the transformation $c_n t^n + \cdots + c_1 t + c_0(\mathrm{id})$ on the same space and where $T$ is a square matrix then $f(T)$ is the matrix $c_n T^n + \cdots + c_1 T + c_0 I$.

**1.4 Remark** If, for instance, $f(x) = x - 3$, then most authors write in the identity matrix: $f(T) = T - 3I$. But most authors don't write in the identity map: $f(t) = t - 3$. In this book we shall also observe this convention.

Of course, if $T = \mathrm{Rep}_{B,B}(t)$ then $f(T) = \mathrm{Rep}_{B,B}(f(t))$, which follows from the relationships $T^j = \mathrm{Rep}_{B,B}(t^j)$, and $cT = \mathrm{Rep}_{B,B}(ct)$, and $T_1 + T_2 = \mathrm{Rep}_{B,B}(t_1 + t_2)$.

As Example 1.2 shows, there may be polynomials of degree smaller than $n^2$ that zero the map or matrix.

**1.5 Definition** The *minimal polynomial* $m(x)$ of a transformation $t$ or a square matrix $T$ is the polynomial of least degree and with leading coefficient 1 such that $m(t)$ is the zero map or $m(T)$ is the zero matrix.

A minimal polynomial always exists by the observation opening this subsection. A minimal polynomial is unique by the 'with leading coefficient 1' clause. This is because if there are two polynomials $m(x)$ and $\hat{m}(x)$ that are both of the minimal degree to make the map or matrix zero (and thus are of equal degree), and both have leading 1's, then their difference $m(x) - \hat{m}(x)$ has a smaller degree than either and still sends the map or matrix to zero. Thus $m(x) - \hat{m}(x)$ is the zero polynomial and the two are equal. (The leading coefficient requirement also prevents a minimal polynomial from being the zero polynomial.)

**1.6 Example** We can see that $m(x) = x^2 - 2x - 1$ is minimal for the matrix of Example 1.2 by computing the powers of $T$ up to the power $n^2 = 4$.

$$T^2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \quad T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad T^4 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

Next, put $c_4 T^4 + c_3 T^3 + c_2 T^2 + c_1 T + c_0 I$ equal to the zero matrix

$$\begin{array}{rcl}
-(1/2)c_4 \qquad + \quad (1/2)c_2 + (\sqrt{3}/2)c_1 + c_0 &=& 0 \\
-(\sqrt{3}/2)c_4 - c_3 - (\sqrt{3}/2)c_2 - \quad (1/2)c_1 \qquad &=& 0 \\
(\sqrt{3}/2)c_4 + c_3 + (\sqrt{3}/2)c_2 + \quad (1/2)c_1 \qquad &=& 0 \\
-(1/2)c_4 \qquad + \quad (1/2)c_2 + (\sqrt{3}/2)c_1 + c_0 &=& 0
\end{array}$$

and use Gauss' method.

$$\begin{array}{rcl}
c_4 \quad - \quad c_2 - \sqrt{3}c_1 - \quad 2c_0 &=& 0 \\
c_3 + \sqrt{3}c_2 + \quad 2c_1 + \sqrt{3}c_0 &=& 0
\end{array}$$

Setting $c_4$, $c_3$, and $c_2$ to zero forces $c_1$ and $c_0$ to also come out as zero. To get a leading one, the most we can do is to set $c_4$ and $c_3$ to zero. Thus the minimal polynomial is quadratic.

Using the method of that example to find the minimal polynomial of a $3\times 3$ matrix would mean doing Gaussian reduction on a system with nine equations in ten unknowns. We shall develop an alternative. To begin, note that we can break a polynomial of a map or a matrix into its components.

**1.7 Lemma** Suppose that the polynomial $f(x) = c_n x^n + \cdots + c_1 x + c_0$ factors as $k(x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$. If $t$ is a linear transformation then these two are equal maps.

$$c_n t^n + \cdots + c_1 t + c_0 = k\cdot(t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$$

Consequently, if $T$ is a square matrix then $f(T)$ and $k\cdot(T-\lambda_1 I)^{q_1}\cdots(T-\lambda_\ell I)^{q_\ell}$ are equal matrices.

Proof. This argument is by induction on the degree of the polynomial. The cases where the polynomial is of degree 0 and 1 are clear. The full induction argument is Exercise 1.7 but the degree two case gives its sense.

A quadratic polynomial factors into two linear terms $f(x) = k(x-\lambda_1)\cdot(x-\lambda_2) = k(x^2+(\lambda_1+\lambda_2)x+\lambda_1\lambda_2)$ (the roots $\lambda_1$ and $\lambda_2$ might be equal). We can check that substituting $t$ for $x$ in the factored and unfactored versions gives the same map.

$$\begin{aligned}\big(k\cdot(t-\lambda_1)\circ(t-\lambda_2)\big)\,(\vec{v}) &= \big(k\cdot(t-\lambda_1)\big)\,(t(\vec{v})-\lambda_2\vec{v}) \\ &= k\cdot\big(t(t(\vec{v}))-t(\lambda_2\vec{v})-\lambda_1 t(\vec{v})-\lambda_1\lambda_2\vec{v}\big) \\ &= k\cdot\big(t\circ t\,(\vec{v})-(\lambda_1+\lambda_2)t(\vec{v})+\lambda_1\lambda_2\vec{v}\big) \\ &= k\cdot(t^2-(\lambda_1+\lambda_2)t+\lambda_1\lambda_2)\,(\vec{v})\end{aligned}$$

The third equality holds because the scalar $\lambda_2$ comes out of the second term, as $t$ is linear. QED

In particular, if a minimial polynomial $m(x)$ for a transformation $t$ factors as $m(x) = (x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$ then $m(t) = (t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t-\lambda_i$ sends some nonzero vectors to zero. So, too, in the matrix case—if $m$ is minimal for $T$ then $m(T) = (T-\lambda_1 I)^{q_1}\cdots(T-\lambda_\ell I)^{q_\ell}$ is the zero matrix and at least one of the matrices $T-\lambda_i I$ sends some nonzero vectors to zero. Rewording both cases: at least some of the $\lambda_i$ are eigenvalues. (See Exercise 29.)

Recall how we have earlier found eigenvalues. We have looked for $\lambda$ such that $T\vec{v} = \lambda\vec{v}$ by considering the equation $\vec{0} = T\vec{v}-x\vec{v} = (T-xI)\vec{v}$ and computing the determinant of the matrix $T-xI$. That determinant is a polynomial in $x$, the characteristic polynomial, whose roots are the eigenvalues. The major result of this subsection, the next result, is that there is a connection between this characteristic polynomial and the minimal polynomial. This results expands on the prior paragraph's insight that some roots of the minimal polynomial are eigenvalues by asserting that every root of the minimal polynomial is an eigenvalue and further that every eigenvalue is a root of the minimal polynomial (this is because it says '$1\leq q_i$' and not just '$0\leq q_i$').

**1.8 Theorem (Cayley-Hamilton)** If the characteristic polynomial of a transformation or square matrix factors into

$$k \cdot (x-\lambda_1)^{p_1}(x-\lambda_2)^{p_2}\cdots(x-\lambda_\ell)^{p_\ell}$$

then its minimal polynomial factors into

$$(x-\lambda_1)^{q_1}(x-\lambda_2)^{q_2}\cdots(x-\lambda_\ell)^{q_\ell}$$

where $1 \leq q_i \leq p_i$ for each $i$ between 1 and $\ell$.

The proof takes up the next three lemmas. Although they are stated only in matrix terms, they apply equally well to maps. We give the matrix version only because it is convenient for the first proof.

The first result is the key — some authors call it the Cayley-Hamilton Theorem and call Theorem 1.8 above a corollary. For the proof, observe that a matrix of polynomials can be thought of as a polynomial with matrix coefficients.

$$\begin{pmatrix} 2x^2+3x-1 & x^2+2 \\ 3x^2+4x+1 & 4x^2+x+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} x^2 + \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

**1.9 Lemma** If $T$ is a square matrix with characteristic polynomial $c(x)$ then $c(T)$ is the zero matrix.

Proof. Let $C$ be $T - xI$, the matrix whose determinant is the characteristic polynomial $c(x) = c_n x^n + \cdots + c_1 x + c_0$.

$$C = \begin{pmatrix} t_{1,1}-x & t_{1,2} & \ldots & \\ t_{2,1} & t_{2,2}-x & & \\ \vdots & & \ddots & \\ & & & t_{n,n}-x \end{pmatrix}$$

Recall that the product of the adjoint of a matrix with the matrix itself is the determinant of that matrix times the identity.

$$c(x)\cdot I = \operatorname{adj}(C)C = \operatorname{adj}(C)(T-xI) = \operatorname{adj}(C)T - \operatorname{adj}(C)\cdot x \qquad (*)$$

The entries of $\operatorname{adj}(C)$ are polynomials, each of degree at most $n-1$ since the minors of a matrix drop a row and column. Rewrite it, as suggested above, as $\operatorname{adj}(C) = C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$ where each $C_i$ is a matrix of scalars. The left and right ends of equation $(*)$ above give this.

$$\begin{aligned} c_n I x^n + c_{n-1} I x^{n-1} + \cdots + c_1 I x + c_0 I = {} & (C_{n-1}T)x^{n-1} + \cdots + (C_1 T)x + C_0 T \\ & - C_{n-1}x^n - C_{n-2}x^{n-1} - \cdots - C_0 x \end{aligned}$$

Equate the coefficients of $x^n$, the coefficients of $x^{n-1}$, etc.

$$\begin{aligned} c_n I &= -C_{n-1} \\ c_{n-1} I &= -C_{n-2} + C_{n-1} T \\ &\vdots \\ c_1 I &= -C_0 + C_1 T \\ c_0 I &= C_0 T \end{aligned}$$

Multiply (from the right) both sides of the first equation by $T^n$, both sides of the second equation by $T^{n-1}$, etc. Add. The result on the left is $c_n T^n + c_{n-1} T^{n-1} + \cdots + c_0 I$, and the result on the right is the zero matrix. QED

We sometimes refer to that lemma by saying that a matrix or map *satisfies* its characteristic polynomial.

**1.10 Lemma** Where $f(x)$ is a polynomial, if $f(T)$ is the zero matrix then $f(x)$ is divisible by the minimal polynomial of $T$. That is, any polynomial satisfied by $T$ is divisable by $T$'s minimal polynomial.

Proof. Let $m(x)$ be minimal for $T$. The Division Theorem for Polynomials gives $f(x) = q(x)m(x) + r(x)$ where the degree of $r$ is strictly less than the degree of $m$. Plugging $T$ in shows that $r(T)$ is the zero matrix, because $T$ satisfies both $f$ and $m$. That contradicts the minimality of $m$ unless $r$ is the zero polynomial. QED

Combining the prior two lemmas gives that the minimal polynomial divides the characteristic polynomial. Thus, any root of the minimal polynomial is also a root of the characteristic polynomial. That is, so far we have that if $m(x) = (x - \lambda_1)^{q_1} \ldots (x - \lambda_i)^{q_i}$ then $c(x)$ must has the form $(x - \lambda_1)^{p_1} \ldots (x - \lambda_i)^{p_i} (x - \lambda_{i+1})^{p_{i+1}} \ldots (x - \lambda_\ell)^{p_\ell}$ where each $q_j$ is less than or equal to $p_j$. The proof of the Cayley-Hamilton Theorem is finished by showing that in fact the characteristic polynomial has no extra roots $\lambda_{i+1}$, etc.

**1.11 Lemma** Each linear factor of the characteristic polynomial of a square matrix is also a linear factor of the minimal polynomial.

Proof. Let $T$ be a square matrix with minimal polynomial $m(x)$ and assume that $x - \lambda$ is a factor of the characteristic polynomial of $T$, that is, assume that $\lambda$ is an eigenvalue of $T$. We must show that $x - \lambda$ is a factor of $m$, that is, that $m(\lambda) = 0$.

In general, where $\lambda$ is associated with the eigenvector $\vec{v}$, for any polynomial function $f(x)$, application of the matrix $f(T)$ to $\vec{v}$ equals the result of multiplying $\vec{v}$ by the scalar $f(\lambda)$. (For instance, if $T$ has eigenvalue $\lambda$ associated with the eigenvector $\vec{v}$ and $f(x) = x^2 + 2x + 3$ then $(T^2 + 2T + 3)(\vec{v}) = T^2(\vec{v}) + 2T(\vec{v}) + 3\vec{v} = \lambda^2 \cdot \vec{v} + 2\lambda \cdot \vec{v} + 3 \cdot \vec{v} = (\lambda^2 + 2\lambda + 3) \cdot \vec{v}$.) Now, as $m(T)$ is the zero matrix, $\vec{0} = m(T)(\vec{v}) = m(\lambda) \cdot \vec{v}$ and therefore $m(\lambda) = 0$. QED

**1.12 Example** We can use the Cayley-Hamilton Theorem to help find the minimal polynomial of this matrix.

$$T=\begin{pmatrix}2&0&0&1\\1&2&0&2\\0&0&2&-1\\0&0&0&1\end{pmatrix}$$

First, its characteristic polynomial $c(x)=(x-1)(x-2)^3$ can be found with the usual determinant. Now, the Cayley-Hamilton Theorem says that $T$'s minimal polynomial is either $(x-1)(x-2)$ or $(x-1)(x-2)^2$ or $(x-1)(x-2)^3$. We can decide among the choices just by computing:

$$(T-1I)(T-2I)=\begin{pmatrix}1&0&0&1\\1&1&0&2\\0&0&1&-1\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\1&0&0&2\\0&0&0&-1\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\1&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

and

$$(T-1I)(T-2I)^2=\begin{pmatrix}0&0&0&0\\1&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\1&0&0&2\\0&0&0&-1\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

and so $m(x)=(x-1)(x-2)^2$.

## Exercises

✓ **1.13** What are the possible minimal polynomials if a matrix has the given characteristic polynomial?
**(a)** $8\cdot(x-3)^4$ **(b)** $(1/3)\cdot(x+1)^3(x-4)$ **(c)** $-1\cdot(x-2)^2(x-5)^2$
**(d)** $5\cdot(x+3)^2(x-1)(x-2)^2$
What is the degree of each possibility?

✓ **1.14** Find the minimal polynomial of each matrix.
**(a)** $\begin{pmatrix}3&0&0\\1&3&0\\0&0&4\end{pmatrix}$ **(b)** $\begin{pmatrix}3&0&0\\1&3&0\\0&0&3\end{pmatrix}$ **(c)** $\begin{pmatrix}3&0&0\\1&3&0\\0&1&3\end{pmatrix}$ **(d)** $\begin{pmatrix}2&0&1\\0&6&2\\0&0&2\end{pmatrix}$
**(e)** $\begin{pmatrix}2&2&1\\0&6&2\\0&0&2\end{pmatrix}$ **(f)** $\begin{pmatrix}-1&4&0&0&0\\0&3&0&0&0\\0&-4&-1&0&0\\3&-9&-4&2&-1\\1&5&4&1&4\end{pmatrix}$

**1.15** Find the minimal polynomial of this matrix.

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

✓ **1.16** What is the minimal polynomial of the differentiation operator $d/dx$ on $\mathcal{P}_n$?

✓ **1.17** Find the minimal polynomial of matrices of this form

$$\begin{pmatrix} \lambda & 0 & 0 & \ldots & & 0 \\ 1 & \lambda & 0 & & & 0 \\ 0 & 1 & \lambda & & & \\ & & & \ddots & & \\ & & & & \lambda & 0 \\ 0 & 0 & \ldots & & 1 & \lambda \end{pmatrix}$$

where the scalar $\lambda$ is fixed (i.e., is not a variable).

**1.18** What is the minimal polynomial of the transformation of $\mathcal{P}_n$ that sends $p(x)$ to $p(x+1)$?

**1.19** What is the minimal polynomial of the map $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates?

**1.20** Find a $3\times 3$ matrix whose minimal polynomial is $x^2$.

**1.21** What is wrong with this claimed proof of Lemma 1.9: "if $c(x) = |T - xI|$ then $c(T) = |T - TI| = 0$"? [Cullen]

**1.22** Verify Lemma 1.9 for $2\times 2$ matrices by direct calculation.

✓ **1.23** Prove that the minimal polynomial of an $n \times n$ matrix has degree at most $n$ (not $n^2$ as might be guessed from this subsection's opening). Verify that this maximum, $n$, can happen.

✓ **1.24** The only eigenvalue of a nilpotent map is zero. Show that the converse statement holds.

**1.25** What is the minimal polynomial of a zero map or matrix? Of an identity map or matrix?

✓ **1.26** Interpret the minimal polynomial of Example 1.2 geometrically.

**1.27** What is the minimal polynomial of a diagonal matrix?

✓ **1.28** A *projection* is any transformation $t$ such that $t^2 = t$. (For instance, the transformation of the plane $\mathbb{R}^2$ projecting each vector onto its first coordinate will, if done twice, result in the same value as if it is done just once.) What is the minimal polynomial of a projection?

**1.29** *The first two items of this question are review.*

**(a)** Prove that the composition of one-to-one maps is one-to-one.

**(b)** Prove that if a linear map is not one-to-one then at least one nonzero vector from the domain is sent to the zero vector in the codomain.

**(c)** Verify the statement, excerpted here, that preceeds Theorem 1.8.

> ...if a minimial polynomial $m(x)$ for a transformation $t$ factors as $m(x) = (x-\lambda_1)^{q_1}\cdots(x-\lambda_\ell)^{q_\ell}$ then $m(t) = (t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_\ell)^{q_\ell}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t-\lambda_i$ sends some nonzero vectors to zero. ...Rewording ...: at least some of the $\lambda_i$ are eigenvalues.

**1.30** True or false: for a transformation on an $n$ dimensional space, if the minimal polynomial has degree $n$ then the map is diagonalizable.

**1.31** Let $f(x)$ be a polynomial. Prove that if $A$ and $B$ are similar matrices then $f(A)$ is similar to $f(B)$.

**(a)** Now show that similar matrices have the same characteristic polynomial.

**(b)** Show that similar matrices have the same minimal polynomial.

**(c)** Decide if these are similar.

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$$

**1.32** **(a)** Show that a matrix is invertible if and only if the constant term in its minimal polynomial is not 0.

**(b)** Show that if a square matrix $T$ is not invertible then there is a nonzero matrix $S$ such that $ST$ and $TS$ both equal the zero matrix.

✓ **1.33** **(a)** Finish the proof of Lemma 1.7.

**(b)** Give an example to show that the result does not hold if $t$ is not linear.

**1.34** Any transformation or square matrix has a minimal polynomial. Does the converse hold?

## IV.2 Jordan Canonical Form

This subsection moves from the canonical form for nilpotent matrices to the one for all matrices.

We have shown that if a map is nilpotent then all of its eigenvalues are zero. We can now prove the converse.

**2.1 Lemma** A linear transformation whose only eigenvalue is zero is nilpotent.

Proof. If a transformation $t$ on an $n$-dimensional space has only the single eigenvalue of zero then its characteristic polynomial is $x^n$. The Cayley-Hamilton Theorem says that a map satisfies its characteristic polynimial so $t^n$ is the zero map. Thus $t$ is nilpotent. QED

We have a canonical form for nilpotent matrices, that is, for each matrix whose single eigenvalue is zero: each such matrix is similar to one that is all zeroes except for blocks of subdiagonal ones. (To make this representation unique we can fix some arrangement of the blocks, say, from longest to shortest.) We next extend this to all single-eigenvalue matrices.

Observe that if $t$'s only eigenvalue is $\lambda$ then $t-\lambda$'s only eigenvalue is 0 because $t(\vec{v}) = \lambda\vec{v}$ if and only if $(t-\lambda)\,(\vec{v}) = 0\cdot\vec{v}$. The natural way to extend the results for nilpotent matrices is to represent $t-\lambda$ in the canonical form $N$, and try to use that to get a simple representation $T$ for $t$. The next result says that this try works.

**2.2 Lemma** If the matrices $T-\lambda I$ and $N$ are similar then $T$ and $N+\lambda I$ are also similar, via the same change of basis matrices.

Proof. With $N = P(T-\lambda I)P^{-1} = PTP^{-1} - P(\lambda I)P^{-1}$ we have $N = PTP^{-1} - PP^{-1}(\lambda I)$ since the diagonal matrix $\lambda I$ commutes with anything, and so $N = PTP^{-1} - \lambda I$. Therefore $N + \lambda I = PTP^{-1}$, as required. QED

**2.3 Example** The characteristic polynomial of

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

is $(x-3)^2$ and so $T$ has only the single eigenvalue 3. Thus for

$$T - 3I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

the only eigenvalue is 0, and $T - 3I$ is nilpotent. The null spaces are routine to find; to ease this computation we take $T$ to represent the transformation $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis (we shall maintain this convention for the rest of the chapter).

$$\mathscr{N}(t-3) = \{\begin{pmatrix} -y \\ y \end{pmatrix} \mid y \in \mathbb{C}\} \qquad \mathscr{N}((t-3)^2) = \mathbb{C}^2$$

The dimensions of these null spaces show that the action of an associated map $t-3$ on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. Thus, the canonical form for $t-3$ with one choice for a string basis is

$$\mathrm{Rep}_{B,B}(t-3) = N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \rangle$$

and by Lemma 2.2, $T$ is similar to this matrix.

$$\mathrm{Rep}_t(B,B) = N + 3I = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

We can produce the similarity computation. Recall from the Nilpotence section how to find the change of basis matrices $P$ and $P^{-1}$ to express $N$ as $P(T-3I)P^{-1}$. The similarity diagram

$$\begin{array}{ccc} \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} & \xrightarrow[T-3I]{t-3} & \mathbb{C}^2_{\text{w.r.t. } \mathcal{E}_2} \\ \text{id}\big\downarrow P & & \text{id}\big\downarrow P \\ C^2_{\text{w.r.t. } B} & \xrightarrow[N]{t-3} & \mathbb{C}^2_{\text{w.r.t. } B} \end{array}$$

describes that to move from the lower left to the upper left we multiply by

$$P^{-1} = \bigl(\mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id})\bigr)^{-1} = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

and to move from the upper right to the lower right we multiply by this matrix.

$$P = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix}$$

So the similarity is expressed by

$$\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

which is easily checked.

**2.4 Example** This matrix has characteristic polynomial $(x-4)^4$

$$T = \begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$$

and so has the single eigenvalue 4. The nullities of $t-4$ are: the null space of $t-4$ has dimension two, the null space of $(t-4)^2$ has dimension three, and the null space of $(t-4)^3$ has dimension four. Thus, $t-4$ has the action on a string basis of $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$ and $\vec{\beta}_4 \mapsto \vec{0}$. This gives the canonical form $N$ for $t-4$, which in turn gives the form for $t$.

$$N + 4I = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

An array that is all zeroes, except for some number $\lambda$ down the diagonal and blocks of subdiagonal ones, is a *Jordan block*. We have shown that Jordan block matrices are canonical representatives of the similarity classes of single-eigenvalue matrices.

**2.5 Example** The $3 \times 3$ matrices whose only eigenvalue is $1/2$ separate into three similarity classes. The three classes have these canonical representatives.

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

In particular, this matrix

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

belongs to the similarity class represented by the middle one, because we have adopted the convention of ordering the blocks of subdiagonal ones from the longest block to the shortest.

We will now finish the program of this chapter by extending this work to cover maps and matrices with multiple eigenvalues. The best possibility for general maps and matrices would be if we could break them into a part involving their first eigenvalue $\lambda_1$ (which we represent using its Jordan block), a part with $\lambda_2$, etc.

This ideal is in fact what happens. For any transformation $t\colon V \to V$, we shall break the space $V$ into the direct sum of a part on which $t-\lambda_1$ is nilpotent, plus a part on which $t-\lambda_2$ is nilpotent, etc. More precisely, we shall take three steps to get to this section's major theorem and the third step shows that $V = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_\ell)$ where $\lambda_1, \ldots, \lambda_\ell$ are $t$'s eigenvalues.

Suppose that $t\colon V \to V$ is a linear transformation. Note that the restriction* of $t$ to a subspace $M$ need not be a linear transformation on $M$ because there may be an $\vec{m} \in M$ with $t(\vec{m}) \notin M$. To ensure that the restriction of a transformation to a 'part' of a space is a transformation on the partwe need the next condition.

**2.6 Definition** Let $t\colon V \to V$ be a transformation. A subspace $M$ is *$t$ invariant* if whenever $\vec{m} \in M$ then $t(\vec{m}) \in M$ (shorter: $t(M) \subseteq M$).

Two examples are that the generalized null space $\mathscr{N}_\infty(t)$ and the generalized range space $\mathscr{R}_\infty(t)$ of any transformation $t$ are invariant. For the generalized null space, if $\vec{v} \in \mathscr{N}_\infty(t)$ then $t^n(\vec{v}) = \vec{0}$ where $n$ is the dimension of the underlying space and so $t(\vec{v}) \in \mathscr{N}_\infty(t)$ because $t^n(\,t(\vec{v})\,)$ is zero also. For the generalized range space, if $\vec{v} \in \mathscr{R}_\infty(t)$ then $\vec{v} = t^n(\vec{w})$ for some $\vec{w}$ and then $t(\vec{v}) = t^{n+1}(\vec{w}) = t^n(\,t(\vec{w})\,)$ shows that $t(\vec{v})$ is also a member of $\mathscr{R}_\infty(t)$.

Thus the spaces $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_i$ invariant. Observe also that $t-\lambda_i$ is nilpotent on $\mathscr{N}_\infty(t-\lambda_i)$ because, simply, if $\vec{v}$ has the property that some power of $t-\lambda_i$ maps it to zero — that is, if it is in the generalized null space — then some power of $t-\lambda_i$ maps it to zero. The generalized null space $\mathscr{N}_\infty(t-\lambda_i)$ is a 'part' of the space on which the action of $t-\lambda_i$ is easy to understand.

The next result is the first of our three steps. It establishes that $t-\lambda_j$ leaves $t-\lambda_i$'s part unchanged.

**2.7 Lemma** A subspace is $t$ invariant if and only if it is $t-\lambda$ invariant for any scalar $\lambda$. In particular, where $\lambda_i$ is an eigenvalue of a linear transformation $t$, then for any other eigenvalue $\lambda_j$, the spaces $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_j$ invariant.

Proof. For the first sentence we check the two implications of the 'if and only if' separately. One of them is easy: if the subspace is $t-\lambda$ invariant for any $\lambda$ then taking $\lambda = 0$ shows that it is $t$ invariant. For the other implication suppose that the subspace is $t$ invariant, so that if $\vec{m} \in M$ then $t(\vec{m}) \in M$, and let $\lambda$ be any scalar. The subspace $M$ is closed under linear combinations and so if $t(\vec{m}) \in M$ then $t(\vec{m}) - \lambda\vec{m} \in M$. Thus if $\vec{m} \in M$ then $(t-\lambda)\,(\vec{m}) \in M$, as required.

The second sentence follows straight from the first. Because the two spaces are $t-\lambda_i$ invariant, they are therefore $t$ invariant. From this, applying the first sentence again, we conclude that they are also $t-\lambda_j$ invariant. QED

* More information on restrictions of functions is in the appendix.

The second step of the three that we will take to prove this section's major result makes use of an additional property of $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$, that they are complementary. Recall that if a space is the direct sum of two others $V=\mathscr{N}\oplus\mathscr{R}$ then any vector $\vec{v}$ in the space breaks into two parts $\vec{v}=\vec{n}+\vec{r}$ where $\vec{n}\in\mathscr{N}$ and $\vec{r}\in\mathscr{R}$, and recall also that if $B_{\mathscr{N}}$ and $B_{\mathscr{R}}$ are bases for $\mathscr{N}$ and $\mathscr{R}$ then the concatenation $B_{\mathscr{N}}{}^\frown B_{\mathscr{R}}$ is linearly independent (and so the two parts of $\vec{v}$ do not "overlap"). The next result says that for any subspaces $\mathscr{N}$ and $\mathscr{R}$ that are complementary as well as $t$ invariant, the action of $t$ on $\vec{v}$ breaks into the "non-overlapping" actions of $t$ on $\vec{n}$ and on $\vec{r}$.

**2.8 Lemma** Let $t\colon V\to V$ be a transformation and let $\mathscr{N}$ and $\mathscr{R}$ be $t$ invariant complementary subspaces of $V$. Then $t$ can be represented by a matrix with blocks of square submatrices $T_1$ and $T_2$

$$\left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right)\begin{array}{l}\}\dim(\mathscr{N})\text{-many rows}\\ \}\dim(\mathscr{R})\text{-many rows}\end{array}$$

where $Z_1$ and $Z_2$ are blocks of zeroes.

Proof. Since the two subspaces are complementary, the concatenation of a basis for $\mathscr{N}$ and a basis for $\mathscr{R}$ makes a basis $B=\langle\vec{\nu}_1,\dots,\vec{\nu}_p,\vec{\mu}_1,\dots,\vec{\mu}_q\rangle$ for $V$. We shall show that the matrix

$$\operatorname{Rep}_{B,B}(t)=\left(\begin{array}{c|c|c} \vdots & & \vdots \\ \operatorname{Rep}_B(t(\vec{\nu}_1)) & \cdots & \operatorname{Rep}_B(t(\vec{\mu}_q)) \\ \vdots & & \vdots \end{array}\right)$$

has the desired form.

Any vector $\vec{v}\in V$ is in $\mathscr{N}$ if and only if its final $q$ components are zeroes when it is represented with respect to $B$. As $\mathscr{N}$ is $t$ invariant, each of the vectors $\operatorname{Rep}_B(t(\vec{\nu}_1))$, ..., $\operatorname{Rep}_B(t(\vec{\nu}_p))$ has that form. Hence the lower left of $\operatorname{Rep}_{B,B}(t)$ is all zeroes.

The argument for the upper right is similar. QED

To see that $t$ has been decomposed into its action on the parts, observe that the restrictions of $t$ to the subspaces $\mathscr{N}$ and $\mathscr{R}$ are represented, with respect to the obvious bases, by the matrices $T_1$ and $T_2$. So, with subspaces that are invariant and complementary, we can split the problem of examining a linear transformation into two lower-dimensional subproblems. The next result illustrates this decomposition into blocks.

**2.9 Lemma** If $T$ is a matrices with square submatrices $T_1$ and $T_2$

$$T=\left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right)$$

where the $Z$'s are blocks of zeroes, then $|T|=|T_1|\cdot|T_2|$.

PROOF. Suppose that $T$ is $n \times n$, that $T_1$ is $p \times p$, and that $T_2$ is $q \times q$. In the permutation formula for the determinant

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

each term comes from a rearrangement of the column numbers $1, \ldots, n$ into a new order $\phi(1), \ldots, \phi(n)$. The upper right block $Z_2$ is all zeroes, so if a $\phi$ has at least one of $p+1, \ldots, n$ among its first $p$ column numbers $\phi(1), \ldots, \phi(p)$ then the term arising from $\phi$ is zero, e.g., if $\phi(1) = n$ then $t_{1,\phi(1)} t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0 \cdot t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0$.

So the above formula reduces to a sum over all permutations with two halves: any significant $\phi$ is the composition of a $\phi_1$ that rearranges only $1, \ldots, p$ and a $\phi_2$ that rearranges only $p+1, \ldots, p+q$. Now, the distributive law (and the fact that the signum of a composition is the product of the signums) gives that this

$$|T_1| \cdot |T_2| = \Bigl( \sum_{\substack{\text{perms } \phi_1 \\ \text{of } 1,\ldots,p}} t_{1,\phi_1(1)} \cdots t_{p,\phi_1(p)} \operatorname{sgn}(\phi_1) \Bigr) \cdot \Bigl( \sum_{\substack{\text{perms } \phi_2 \\ \text{of } p+1,\ldots,p+q}} t_{p+1,\phi_2(p+1)} \cdots t_{p+q,\phi_2(p+q)} \operatorname{sgn}(\phi_2) \Bigr)$$

equals $|T| = \sum_{\text{significant } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$. QED

**2.10 Example**

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 36$$

From Lemma 2.9 we conclude that if two subspaces are complementary and $t$ invariant then $t$ is nonsingular if and only if its restrictions to both subspaces are nonsingular.

Now for the promised third, final, step to the main result.

**2.11 Lemma** If a linear transformation $t\colon V \to V$ has the characteristic polynomial $(x - \lambda_1)^{p_1} \ldots (x - \lambda_\ell)^{p_\ell}$ then (1) $V = \mathscr{N}_\infty(t - \lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t - \lambda_\ell)$ and (2) $\dim(\mathscr{N}_\infty(t - \lambda_i)) = p_i$.

PROOF. Because $\dim(V)$ is the degree $p_1 + \cdots + p_\ell$ of the characteristic polynomial, to establish statement (1) we need only show that statement (2) holds and that $\mathscr{N}_\infty(t - \lambda_i) \cap \mathscr{N}_\infty(t - \lambda_j)$ is trivial whenever $i \neq j$.

For the latter, by Lemma 2.7, both $\mathscr{N}_\infty(t - \lambda_i)$ and $\mathscr{N}_\infty(t - \lambda_j)$ are $t$ invariant. Notice that an intersection of $t$ invariant subspaces is $t$ invariant and so the restriction of $t$ to $\mathscr{N}_\infty(t - \lambda_i) \cap \mathscr{N}_\infty(t - \lambda_j)$ is a linear transformation. But both $t - \lambda_i$ and $t - \lambda_j$ are nilpotent on this subspace and so if $t$ has any eigenvalues on the intersection then its "only" eigenvalue is both $\lambda_i$ and $\lambda_j$. That cannot be, so this restriction has no eigenvalues: $\mathscr{N}_\infty(t-\lambda_i) \cap \mathscr{N}_\infty(t-\lambda_j)$ is trivial (Lemma 3.10 shows that the only transformation without any eigenvalues is on the trivial space).

To prove statement (2), fix the index $i$. Decompose $V$ as $\mathscr{N}_\infty(t-\lambda_i) \oplus \mathscr{R}_\infty(t-\lambda_i)$ and apply Lemma 2.8.

$$T = \left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right) \begin{array}{l} \}\dim(\,\mathscr{N}_\infty(t-\lambda_i)\,)\text{-many rows} \\ \}\dim(\,\mathscr{R}_\infty(t-\lambda_i)\,)\text{-many rows} \end{array}$$

By Lemma 2.9, $|T - xI| = |T_1 - xI| \cdot |T_2 - xI|$. By the uniqueness clause of the Fundamental Theorem of Arithmetic, the determinants of the blocks have the same factors as the characteristic polynomial $|T_1 - xI| = (x-\lambda_1)^{q_1} \ldots (x-\lambda_\ell)^{q_\ell}$ and $|T_2 - xI| = (x-\lambda_1)^{r_1} \ldots (x-\lambda_\ell)^{r_\ell}$, and the sum of the powers of these factors is the power of the factor in the characteristic polynomial: $q_1 + r_1 = p_1$, $\ldots$, $q_\ell + r_\ell = p_\ell$. Statement (2) will be proved if we will show that $q_i = p_i$ and that $q_j = 0$ for all $j \neq i$, because then the degree of the polynomial $|T_1 - xI|$ — which equals the dimension of the generalized null space — is as required.

For that, first, as the restriction of $t - \lambda_i$ to $\mathscr{N}_\infty(t-\lambda_i)$ is nilpotent on that space, the only eigenvalue of $t$ on it is $\lambda_i$. Thus the characteristic equation of $t$ on $\mathscr{N}_\infty(t-\lambda_i)$ is $|T_1 - xI| = (x-\lambda_i)^{q_i}$. And thus $q_j = 0$ for all $j \neq i$.

Now consider the restriction of $t$ to $\mathscr{R}_\infty(t-\lambda_i)$. By Note II.2.2, the map $t - \lambda_i$ is nonsingular on $\mathscr{R}_\infty(t-\lambda_i)$ and so $\lambda_i$ is not an eigenvalue of $t$ on that subspace. Therefore, $x - \lambda_i$ is not a factor of $|T_2 - xI|$, and so $q_i = p_i$. QED

Our major result just translates those steps into matrix terms.

**2.12 Theorem** Any square matrix is similar to one in *Jordan form*

$$\begin{pmatrix} J_{\lambda_1} & & \textit{–zeroes–} & & \\ & J_{\lambda_2} & & & \\ & & \ddots & & \\ & & & J_{\lambda_{\ell-1}} & \\ & & \textit{–zeroes–} & & J_{\lambda_\ell} \end{pmatrix}$$

where each $J_\lambda$ is the Jordan block associated with the eigenvalue $\lambda$ of the original matrix (that is, is all zeroes except for $\lambda$'s down the diagonal and some subdiagonal ones).

Proof. Given an $n \times n$ matrix $T$, consider the linear map $t\colon \mathbb{C}^n \to \mathbb{C}^n$ that it represents with respect to the standard bases. Use the prior lemma to write $\mathbb{C}^n = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_\ell)$ where $\lambda_1, \ldots, \lambda_\ell$ are the eigenvalues of $t$. Because each $\mathscr{N}_\infty(t-\lambda_i)$ is $t$ invariant, Lemma 2.8 and the prior lemma show that $t$ is represented by a matrix that is all zeroes except for square blocks along the diagonal. To make those blocks into Jordan blocks, pick each $B_{\lambda_i}$ to be a string basis for the action of $t - \lambda_i$ on $\mathscr{N}_\infty(t-\lambda_i)$. QED

Jordan form is a canonical form for similarity classes of square matrices, provided that we make it unique by arranging the Jordan blocks from least eigenvalue to greatest and then arranging the subdiagonal 1 blocks inside each Jordan block from longest to shortest.

**2.13 Example** This matrix has the characteristic polynomial $(x-2)^2(x-6)$.

$$T = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

We will handle the eigenvalues 2 and 6 separately.

Computation of the powers, and the null spaces and nullities, of $T-2I$ is routine. (Recall from Example 2.3 the convention of taking $T$ to represent a transformation, here $t\colon \mathbb{C}^3 \to \mathbb{C}^3$, with respect to the standard basis.)

| *power p* | $(T-2I)^p$ | $\mathscr{N}((t-2)^p)$ | *nullity* |
|---|---|---|---|
| 1 | $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ | $\{\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \mid x \in \mathbb{C}\}$ | 1 |
| 2 | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ | $\{\begin{pmatrix} x \\ -z/2 \\ z \end{pmatrix} \mid x, z \in \mathbb{C}\}$ | 2 |
| 3 | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 64 & 32 \\ 0 & 0 & 0 \end{pmatrix}$ | *–same–* | — |

So the generalized null space $\mathscr{N}_\infty(t-2)$ has dimension two. We've noted that the restriction of $t-2$ is nilpotent on this subspace. From the way that the nullities grow we know that the action of $t-2$ on a string basis $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. Thus the restriction can be represented in the canonical form

$$N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \mathrm{Rep}_{B,B}(t-2) \qquad B_2 = \langle \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} \rangle$$

where many choices of basis are possible. Consequently, the action of the restriction of $t$ to $\mathscr{N}_\infty(t-2)$ is represented by this matrix.

$$J_2 = N_2 + 2I = \mathrm{Rep}_{B_2,B_2}(t) = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

The second eigenvalue's computations are easier. Because the power of $x-6$ in the characteristic polynomial is one, the restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$ must be nilpotent of index one. Its action on a string basis must be $\vec{\beta}_3 \mapsto \vec{0}$ and since it is the zero map, its canonical form $N_6$ is the $1\times 1$ zero matrix. Consequently, the canonical form $J_6$ for the action of $t$ on $\mathscr{N}_\infty(t-6)$ is the $1\times 1$ matrix with the single entry 6. For the basis we can use any nonzero vector from the generalized null space.

$$B_6 = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$$

Taken together, these two give that the Jordan form of $T$ is

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

where $B$ is the concatenation of $B_2$ and $B_6$.

**2.14 Example** Contrast the prior example with

$$T = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

which has the same characteristic polynomial $(x-2)^2(x-6)$.

While the characteristic polynomial is the same,

| *power p* | $(T-2I)^p$ | $\mathscr{N}((t-2)^p)$ | *nullity* |
|---|---|---|---|
| 1 | $\begin{pmatrix} 0 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ | $\{\begin{pmatrix} x \\ -z/2 \\ z \end{pmatrix} \bigm\vert x, z \in \mathbb{C}\}$ | 2 |
| 2 | $\begin{pmatrix} 0 & 8 & 4 \\ 0 & 16 & 8 \\ 0 & 0 & 0 \end{pmatrix}$ | *–same–* | — |

here the action of $t-2$ is stable after only one application — the restriction of of $t-2$ to $\mathscr{N}_\infty(t-2)$ is nilpotent of index only one. (So the contrast with the prior example is that while the characteristic polynomial tells us to look at the action of the $t-2$ on its generalized null space, the characteristic polynomial does not describe completely its action and we must do some computations to find, in this example, that the minimal polynomial is $(x-2)(x-6)$.) The restriction of $t-2$ to the generalized null space acts on a string basis as $\vec{\beta}_1 \mapsto \vec{0}$ and $\vec{\beta}_2 \mapsto \vec{0}$, and we get this Jordan block associated with the eigenvalue 2.

$$J_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

For the other eigenvalue, the arguments for the second eigenvalue of the prior example apply again. The restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$ is nilpotent of index one (it can't be of index less than one, and since $x-6$ is a factor of the characteristic polynomial to the power one it can't be of index more than one either). Thus $t-6$'s canonical form $N_6$ is the $1\times 1$ zero matrix, and the associated Jordan block $J_6$ is the $1\times 1$ matrix with entry 6.

Therefore, $T$ is diagonalizable.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}2&0&0\\0&2&0\\0&0&6\end{pmatrix}\qquad B=B_2\frown B_6=\langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix},\begin{pmatrix}3\\4\\0\end{pmatrix}\rangle$$

(Checking that the third vector in $B$ is in the nullspace of $t-6$ is routine.)

**2.15 Example** A bit of computing with

$$T=\begin{pmatrix}-1&4&0&0&0\\0&3&0&0&0\\0&-4&-1&0&0\\3&-9&-4&2&-1\\1&5&4&1&4\end{pmatrix}$$

shows that its characteristic polynomial is $(x-3)^3(x+1)^2$. This table

| *power p* | $(T-3I)^p$ | $\mathscr{N}((t-3)^p)$ | *nullity* |
|---|---|---|---|
| 1 | $\begin{pmatrix}-4&4&0&0&0\\0&0&0&0&0\\0&-4&-4&0&0\\3&-9&-4&-1&-1\\1&5&4&1&1\end{pmatrix}$ | $\{\begin{pmatrix}-(u+v)/2\\-(u+v)/2\\(u+v)/2\\u\\v\end{pmatrix}\bigm\vert u,v\in\mathbb{C}\}$ | 2 |
| 2 | $\begin{pmatrix}16&-16&0&0&0\\0&0&0&0&0\\0&16&16&0&0\\-16&32&16&0&0\\0&-16&-16&0&0\end{pmatrix}$ | $\{\begin{pmatrix}-z\\-z\\z\\u\\v\end{pmatrix}\bigm\vert z,u,v\in\mathbb{C}\}$ | 3 |
| 3 | $\begin{pmatrix}-64&64&0&0&0\\0&0&0&0&0\\0&-64&-64&0&0\\64&-128&-64&0&0\\0&64&64&0&0\end{pmatrix}$ | *–same–* | — |

shows that the restriction of $t-3$ to $\mathscr{N}_\infty(t-3)$ acts on a string basis via the two strings $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$ and $\vec{\beta}_3\mapsto\vec{0}$.

A similar calculation for the other eigenvalue

| *power* $p$ | $(T+1I)^p$ | $\mathscr{N}((t+1)^p)$ | *nullity* |
|---|---|---|---|
| 1 | $\begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 3 & -9 & -4 & 3 & -1 \\ 1 & 5 & 4 & 1 & 5 \end{pmatrix}$ | $\{\begin{pmatrix} -(u+v) \\ 0 \\ -v \\ u \\ v \end{pmatrix} \bigm\vert u,v\in\mathbb{C}\}$ | 2 |
| 2 | $\begin{pmatrix} 0 & 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 & 0 \\ 8 & -40 & -16 & 8 & -8 \\ 8 & 24 & 16 & 8 & 24 \end{pmatrix}$ | –*same*– | — |

shows that the restriction of $t+1$ to its generalized null space acts on a string basis via the two separate strings $\vec{\beta}_4 \mapsto \vec{0}$ and $\vec{\beta}_5 \mapsto \vec{0}$.

Therefore $T$ is similar to this Jordan form matrix.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

We close with the statement that the subjects considered earlier in this Chpater are indeed, in this sense, exhaustive.

**2.16 Corollary** Every square matrix is similar to the sum of a diagonal matrix and a nilpotent matrix.

## Exercises

**2.17** Do the check for Example 2.3.

**2.18** Each matrix is in Jordan form. State its characteristic polynomial and its minimal polynomial.

**(a)** $\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

**(e)** $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & -4 \end{pmatrix}$ **(g)** $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

**(h)** $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ **(i)** $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

✓ **2.19** Find the Jordan form from the given data.

**(a)** The matrix $T$ is 5×5 with the single eigenvalue 3. The nullities of the powers are: $T-3I$ has nullity two, $(T-3I)^2$ has nullity three, $(T-3I)^3$ has nullity four, and $(T-3I)^4$ has nullity five.

**(b)** The matrix $S$ is $5\times5$ with two eigenvalues. For the eigenvalue 2 the nullities are: $S-2I$ has nullity two, and $(S-2I)^2$ has nullity four. For the eigenvalue $-1$ the nullities are: $S+1I$ has nullity one.

**2.20** Find the change of basis matrices for each example.
**(a)** Example 2.13 **(b)** Example 2.14 **(c)** Example 2.15

✓ **2.21** Find the Jordan form and a Jordan basis for each matrix.

**(a)** $\begin{pmatrix} -10 & 4 \\ -25 & 10 \end{pmatrix}$

**(b)** $\begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}$

**(c)** $\begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{pmatrix}$

**(d)** $\begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

**(e)** $\begin{pmatrix} 9 & 7 & 3 \\ -9 & -7 & -4 \\ 4 & 4 & 4 \end{pmatrix}$

**(f)** $\begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}$

**(g)** $\begin{pmatrix} 7 & 1 & 2 & 2 \\ 1 & 4 & -1 & -1 \\ -2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$

✓ **2.22** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^2(x+2)^2$.

**2.23** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^3(x+2)$.

✓ **2.24** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^3(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

**2.25** Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^4(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

✓ **2.26** Diagonalize these.

**(a)** $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

✓ **2.27** Find the Jordan matrix representing the differentiation operator on $\mathcal{P}_3$.

✓ **2.28** Decide if these two are similar.

$$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

**2.29** Find the Jordan form of this matrix.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Also give a Jordan basis.

**2.30** How many similarity classes are there for $3\times3$ matrices whose only eigenvalues are $-3$ and 4?

✓ **2.31** Prove that a matrix is diagonalizable if and only if its minimal polynomial has only linear factors.

**2.32** Give an example of a linear transformation on a vector space that has no non-trivial invariant subspaces.

**2.33** Show that a subspace is $t-\lambda_1$ invariant if and only if it is $t-\lambda_2$ invariant.

**2.34** Prove or disprove: two $n\times n$ matrices are similar if and only if they have the same characteristic and minimal polynomials.

**2.35** The *trace* of a square matrix is the sum of its diagonal entries.

**(a)** Find the formula for the characteristic polynomial of a $2\times 2$ matrix.

**(b)** Show that trace is invariant under similarity, and so we can sensibly speak of the 'trace of a map'. (*Hint:* see the prior item.)

**(c)** Is trace invariant under matrix equivalence?

**(d)** Show that the trace of a map is the sum of its eigenvalues (counting multiplicities).

**(e)** Show that the trace of a nilpotent map is zero. Does the converse hold?

**2.36** To use Definition 2.6 to check whether a subspace is $t$ invariant, we seemingly have to check all of the infinitely many vectors in a (nontrivial) subspace to see if they satisfy the condition. Prove that a subspace is $t$ invariant if and only if its subbasis has the property that for all of its elements, $t(\vec{\beta})$ is in the subspace.

✓ **2.37** Is $t$ invariance preserved under intersection? Under union? Complementation? Sums of subspaces?

**2.38** Give a way to order the Jordan blocks if some of the eigenvalues are complex numbers. That is, suggest a reasonable ordering for the complex numbers.

**2.39** Let $\mathcal{P}_j(\mathbb{R})$ be the vector space over the reals of degree $j$ polynomials. Show that if $j\leq k$ then $\mathcal{P}_j(\mathbb{R})$ is an invariant subspace of $\mathcal{P}_k(\mathbb{R})$ under the differentiation operator. In $\mathcal{P}_7(\mathbb{R})$, does any of $\mathcal{P}_0(\mathbb{R}),\ \ldots,\ \mathcal{P}_6(\mathbb{R})$ have an invariant complement?

**2.40** In $\mathcal{P}_n(\mathbb{R})$, the vector space (over the reals) of degree $n$ polynomials,

$$\mathcal{E}=\{p(x)\in\mathcal{P}_n(\mathbb{R})\bigm| p(-x)=p(x)\text{ for all }x\}$$

and

$$\mathcal{O}=\{p(x)\in\mathcal{P}_n(\mathbb{R})\bigm| p(-x)=-p(x)\text{ for all }x\}$$

are the *even* and the *odd* polynomials; $p(x)=x^2$ is even while $p(x)=x^3$ is odd. Show that they are subspaces. Are they complementary? Are they invariant under the differentiation transformation?

**2.41** Lemma 2.8 says that if $M$ and $N$ are invariant complements then $t$ has a representation in the given block form (with respect to the same ending as starting basis, of course). Does the implication reverse?

**2.42** A matrix $S$ is the *square root* of another $T$ if $S^2=T$. Show that any nonsingular matrix has a square root.

# Topic: Method of Powers

In practice, calculating eigenvalues and eigenvectors is a difficult problem. Finding, and solving, the characteristic polynomial of the large matrices often encountered in applications is too slow and too hard. Other techniques, indirect ones that avoid the characteristic polynomial, are used. Here we shall see such a method that is suitable for large matrices that are 'sparse' (the great majority of the entries are zero).

Suppose that the $n\times n$ matrix $T$ has the $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\mathbb{R}^n$ has a basis that is composed of the associated eigenvectors $\langle\vec{\zeta}_1, \ldots, \vec{\zeta}_n\rangle$. For any $\vec{v}\in\mathbb{R}^n$, where $\vec{v}=c_1\vec{\zeta}_1+\cdots+c_n\vec{\zeta}_n$, iterating $T$ on $\vec{v}$ gives these.

$$\begin{aligned}
T\vec{v} &= c_1\lambda_1\vec{\zeta}_1+c_2\lambda_2\vec{\zeta}_2+\cdots+c_n\lambda_n\vec{\zeta}_n \\
T^2\vec{v} &= c_1\lambda_1^2\vec{\zeta}_1+c_2\lambda_2^2\vec{\zeta}_2+\cdots+c_n\lambda_n^2\vec{\zeta}_n \\
T^3\vec{v} &= c_1\lambda_1^3\vec{\zeta}_1+c_2\lambda_2^3\vec{\zeta}_2+\cdots+c_n\lambda_n^3\vec{\zeta}_n \\
&\vdots \\
T^k\vec{v} &= c_1\lambda_1^k\vec{\zeta}_1+c_2\lambda_2^k\vec{\zeta}_2+\cdots+c_n\lambda_n^k\vec{\zeta}_n
\end{aligned}$$

If one of the eigenvaluse, say, $\lambda_1$, has a larger absolute value than any of the other eigenvalues then its term will dominate the above expression. Put another way, dividing through by $\lambda_1^k$ gives this,

$$\frac{T^k\vec{v}}{\lambda_1^k}=c_1\vec{\zeta}_1+c_2\frac{\lambda_2^k}{\lambda_1^k}\vec{\zeta}_2+\cdots+c_n\frac{\lambda_n^k}{\lambda_1^k}\vec{\zeta}_n$$

and, because $\lambda_1$ is assumed to have the largest absolute value, as $k$ gets larger the fractions go to zero. Thus, the entire expression goes to $c_1\vec{\zeta}_1$.

That is (as long as $c_1$ is not zero), as $k$ increases, the vectors $T^k\vec{v}$ will tend toward the direction of the eigenvectors associated with the dominant eigenvalue, and, consequently, the ratios of the lengths $\|\,T^k\vec{v}\,\|/\|\,T^{k-1}\vec{v}\,\|$ will tend toward that dominant eigenvalue.

For example (sample computer code for this follows the exercises), because the matrix

$$T=\begin{pmatrix}3&0\\8&-1\end{pmatrix}$$

is triangular, its eigenvalues are just the entries on the diagonal, 3 and $-1$. Arbitrarily taking $\vec{v}$ to have the components 1 and 1 gives

| $\vec{v}$ | $T\vec{v}$ | $T^2\vec{v}$ | $\cdots$ | $T^9\vec{v}$ | $T^{10}\vec{v}$ |
|---|---|---|---|---|---|
| $\begin{pmatrix}1\\1\end{pmatrix}$ | $\begin{pmatrix}3\\7\end{pmatrix}$ | $\begin{pmatrix}9\\17\end{pmatrix}$ | $\cdots$ | $\begin{pmatrix}19\,683\\39\,367\end{pmatrix}$ | $\begin{pmatrix}59\,049\\118\,097\end{pmatrix}$ |

and the ratio between the lengths of the last two is 2.999 9.

Two implementation issues must be addressed. The first issue is that, instead of finding the powers of $T$ and applying them to $\vec{v}$, we will compute $\vec{v}_1$ as $T\vec{v}$ and then compute $\vec{v}_2$ as $T\vec{v}_1$, etc. (i.e., we never separately calculate $T^2$, $T^3$, etc.). These matrix-vector products can be done quickly even if $T$ is large, provided that it is sparse. The second issue is that, to avoid generating numbers that are so large that they overflow our computer's capability, we can normalize the $\vec{v}_i$'s at each step. For instance, we can divide each $\vec{v}_i$ by its length (other possibilities are to divide it by its largest component, or simply by its first component). We thus implement this method by generating

$$\begin{aligned}
\vec{w}_0 &= \vec{v}_0/\|\vec{v}_0\| \\
\vec{v}_1 &= T\vec{w}_0 \\
\vec{w}_1 &= \vec{v}_1/\|\vec{v}_1\| \\
\vec{v}_2 &= T\vec{w}_2 \\
&\vdots \\
\vec{w}_{k-1} &= \vec{v}_{k-1}/\|\vec{v}_{k-1}\| \\
\vec{v}_k &= T\vec{w}_k
\end{aligned}$$

until we are satisfied. Then the vector $\vec{v}_k$ is an approximation of an eigenvector, and the approximation of the dominant eigenvalue is the ratio $\|\vec{v}_k\|/\|\vec{w}_{k-1}\| = \|\vec{v}_k\|$.

One way we could be 'satisfied' is to iterate until our approximation of the eigenvalue settles down. We could decide, for instance, to stop the iteration process not after some fixed number of steps, but instead when $\|\vec{v}_k\|$ differs from $\|\vec{v}_{k-1}\|$ by less than one percent, or when they agree up to the second significant digit.

The rate of convergence is determined by the rate at which the powers of $\|\lambda_2/\lambda_1\|$ go to zero, where $\lambda_2$ is the eigenvalue of second largest norm. If that ratio is much less than one then convergence is fast, but if it is only slightly less than one then convergence can be quite slow. Consequently, the method of powers is not the most commonly used way of finding eigenvalues (although it is the simplest one, which is why it is here as the illustration of the possibility of computing eigenvalues without solving the characteristic polynomial). Instead, there are a variety of methods that generally work by first replacing the given matrix $T$ with another that is similar to it and so has the same eigenvalues, but is in some reduced form such as *tridiagonal form*: the only nonzero entries are on the diagonal, or just above or below it. Then special techniques can be used to find the eigenvalues. Once the eigenvalues are known, the eigenvectors of $T$ can be easily computed. These other methods are outside of our scope. A good reference is [Goult, *et al.*]

## Exercises

**1** Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1 and 2. Compare the answer with the one obtained by solving the characteristic equation.

**(a)** $\begin{pmatrix}1&5\\0&4\end{pmatrix}$ **(b)** $\begin{pmatrix}3&2\\-1&0\end{pmatrix}$

**2** Redo the prior exercise by iterating until $\|\vec{v}_k\| - \|\vec{v}_{k-1}\|$ has absolute value less than 0.01 At each step, normalize by dividing each vector by its length. How many iterations are required? Are the answers significantly different?

**3** Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1, 2, and 3. Compare the answer with the one obtained by solving the characteristic equation.

**(a)** $\begin{pmatrix}4&0&1\\-2&1&0\\-2&0&1\end{pmatrix}$ **(b)** $\begin{pmatrix}-1&2&2\\2&2&2\\-3&-6&-6\end{pmatrix}$

**4** Redo the prior exercise by iterating until $\|\vec{v}_k\| - \|\vec{v}_{k-1}\|$ has absolute value less than 0.01. At each step, normalize by dividing each vector by its length. How many iterations does it take? Are the answers significantly different?

**5** What happens if $c_1 = 0$? That is, what happens if the initial vector does not to have any component in the direction of the relevant eigenvector?

**6** How can the method of powers be adopted to find the smallest eigenvalue?

## Computer Code

This is the code for the computer algebra system Octave that was used to do the calculation above. (It has been lightly edited to remove blank lines, etc.)

```
>T=[3, 0;
    8, -1]
T=
   3   0
   8  -1
>v0=[1; 2]
v0=
   1
   1
>v1=T*v0
v1=
   3
   7
>v2=T*v1
v2=
   9
  17
>T9=T**9
T9=
  19683  0
  39368 -1
>T10=T**10
T10=
  59049  0
 118096  1
>v9=T9*v0
v9=
```

```
  19683
  39367
>v10=T10*v0
v10=
  59049
 118096
>norm(v10)/norm(v9)
ans=2.9999
```

Remark: we are ignoring the power of Octave here; there are built-in functions to automatically apply quite sophisticated methods to find eigenvalues and eigenvectors. Instead, we are using just the system as a calculator.

# Topic: Stable Populations

Imagine a reserve park with animals from a species that we are trying to protect. The park doesn't have a fence and so animals cross the boundary, both from the inside out and in the other direction. Every year, 10% of the animals from inside of the park leave, and 1% of the animals from the outside find their way in. We can ask if we can find a stable level of population for this park: is there a population that, once established, will stay constant over time, with the number of animals leaving equal to the number of animals entering?

To answer that question, we must first establish the equations. Let the year $n$ population in the park be $p_n$ and in the rest of the world be $r_n$.

$$\begin{aligned} p_{n+1} &= .90p_n + .01r_n \\ r_{n+1} &= .10p_n + .99r_n \end{aligned}$$

We can set this system up as a matrix equation (see the Markov Chain topic).

$$\begin{pmatrix} p_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .90 & .01 \\ .10 & .99 \end{pmatrix} \begin{pmatrix} p_n \\ r_n \end{pmatrix}$$

Now, "stable level" means that $p_{n+1} = p_n$ and $r_{n+1} = r_n$, so that the matrix equation $\vec{v}_{n+1} = T\vec{v}_n$ becomes $\vec{v} = T\vec{v}$. We are therefore looking for eigenvectors for $T$ that are associated with the eigenvalue 1. The equation $(I - T)\vec{v} = \vec{0}$ is

$$\begin{pmatrix} .10 & .01 \\ .10 & .01 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenspace: vectors with the restriction that $p = .1r$. Coupled with additional information, that the total world population of this species is is $p + r = 110\,000$, we find that the stable state is $p = 10,000$ and $r = 100,000$.

If we start with a park population of ten thousand animals, so that the rest of the world has one hundred thousand, then every year ten percent (a thousand animals) of those inside will leave the park, and every year one percent (a thousand) of those from the rest of the world will enter the park. It is stable, self-sustaining.

Now imagine that we are trying to gradually build up the total world population of this species. We can try, for instance, to have the world population grow at a rate of 1% per year. In this case, we can take a "stable" state for the park's population to be that it also grows at 1% per year. The equation $\vec{v}_{n+1} = 1.01 \cdot \vec{v}_n = T\vec{v}_n$ leads to $((1.01 \cdot I) - T)\vec{v} = \vec{0}$, which gives this system.

$$\begin{pmatrix} .11 & .01 \\ .10 & .02 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrix is nonsingular, and so the only solution is $p = 0$ and $r = 0$. Thus, there is no (usable) initial population that we can establish at the park and expect that it will grow at the same rate as the rest of the world.

Knowing that an annual world population growth rate of 1% forces an unstable park population, we can ask which growth rates there are that would allow an initial population for the park that will be self-sustaining. We consider $\lambda\vec{v} = T\vec{v}$ and solve for $\lambda$.

$$0 = \begin{vmatrix} \lambda - .9 & .01 \\ .10 & \lambda - .99 \end{vmatrix} = (\lambda - .9)(\lambda - .99) - (.10)(.01) = \lambda^2 - 1.89\lambda + .89$$

A shortcut to factoring that quadratic is our knowledge that $\lambda = 1$ is an eigenvalue of $T$, so the other eigenvalue is .89. Thus there are two ways to have a stable park population (a population that grows at the same rate as the population of the rest of the world, despite the leaky park boundaries): have a world population that is does not grow or shrink, and have a world population that shrinks by 11% every year.

So this is one meaning of eigenvalues and eigenvectors — they give a stable state for a system. If the eigenvalue is 1 then the system is static. If the eigenvalue isn't 1 then the system is either growing or shrinking, but in a dynamically-stable way.

## Exercises

**1** What initial population for the park discussed above should be set up in the case where world populations are allowed to decline by 11% every year?

**2** What will happen to the population of the park in the event of a growth in world population of 1% per year? Will it lag the world growth, or lead it? Assume that the inital park population is ten thousand, and the world population is one hunderd thousand, and calculate over a ten year span.

**3** The park discussed above is partially fenced so that now, every year, only 5% of the animals from inside of the park leave (still, about 1% of the animals from the outside find their way in). Under what conditions can the park maintain a stable population now?

**4** Suppose that a species of bird only lives in Canada, the United States, or in Mexico. Every year, 4% of the Canadian birds travel to the US, and 1% of them travel to Mexico. Every year, 6% of the US birds travel to Canada, and 4% go to Mexico. From Mexico, every year 10% travel to the US, and 0% go to Canada.

**(a)** Give the transition matrix.

**(b)** Is there a way for the three countries to have constant populations?

**(c)** Find all stable situations.

# Topic: Linear Recurrences

In 1202 Leonardo of Pisa, also known as Fibonacci, posed this problem.

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

This moves past an elementary exponential growth model for population increase to include the fact that there is an initial period where newborns are not fertile. However, it retains other simplyfing assumptions, such as that there is no gestation period and no mortality.

The number of newborn pairs that will appear in the upcoming month is simply the number of pairs that were alive last month, since those will all be fertile, having been alive for two months. The number of pairs alive next month is the sum of the number alive last month and the number of newborns.

$$f(n+1) = f(n) + f(n-1) \qquad \text{where } f(0) = 1,\ f(1) = 1$$

The is an example of a *recurrence relation* (it is called that because the values of $f$ are calculated by looking at other, prior, values of $f$). From it, we can easily answer Fibonacci's twelve-month question.

| *month* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *pairs* | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |

The sequence of numbers defined by the above equation (of which the first few are listed) is the *Fibonacci sequence*. The material of this chapter can be used to give a formula with which we can can calculate $f(n+1)$ without having to first find $f(n)$, $f(n-1)$, etc.

For that, observe that the recurrence is a linear relationship and so we can give a suitable matrix formulation of it.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(n) \\ f(n-1) \end{pmatrix} = \begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} \qquad \text{where } \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then, where we write $T$ for the matrix and $\vec{v}_n$ for the vector with components $f(n+1)$ and $f(n)$, we have that $\vec{v}_n = T^n \vec{v}_0$. The advantage of this matrix formulation is that by diagonalizing $T$ we get a fast way to compute its powers: where $T = PDP^{-1}$ we have $T^n = PD^nP^{-1}$, and the $n$-th power of the diagonal matrix $D$ is the diagonal matrix whose entries that are the $n$-th powers of the entries of $D$.

The characteristic equation of $T$ is $\lambda^2 - \lambda - 1$. The quadratic formula gives its roots as $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. Diagonalizing gives this.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

Introducing the vectors and taking the $n$-th power, we have

$$\begin{aligned}\begin{pmatrix} f(n+1) \\ f(n) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2}^n & 0 \\ 0 & \frac{1-\sqrt{5}}{2}^n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix}\end{aligned}$$

We can compute $f(n)$ from the second component of that equation.

$$f(n) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right]$$

Notice that $f$ is dominated by its first term because $(1-\sqrt{5})/2$ is less than one, so its powers go to zero. Although we have extended the elementary model of population growth by adding a delay period before the onset of fertility, we nonetheless still get an (asmyptotically) exponential function.

In general, a *linear recurrence relation* has the form

$$f(n+1) = a_n f(n) + a_{n-1} f(n-1) + \cdots + a_{n-k} f(n-k)$$

(it is also called a *difference equation*). This recurrence relation is *homogeneous* because there is no constant term; i.e, it can be put into the form $0 = -f(n+1) + a_n f(n) + a_{n-1} f(n-1) + \cdots + a_{n-k} f(n-k)$. This is said to be a relation of *order* $k$. The relation, along with the *initial conditions* $f(0)$, ..., $f(k)$ completely determine a sequence. For instance, the Fibonacci relation is of order 2 and it, along with the two initial conditions $f(0) = 1$ and $f(1) = 1$, determines the Fibonacci sequence simply because we can compute any $f(n)$ by first computing $f(2)$, $f(3)$, etc. In this Topic, we shall see how linear algebra can be used to solve linear recurrence relations.

First, we define the vector space in which we are working. Let $V$ be the set of functions $f$ from the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ to the real numbers. (Below we shall have functions with domain $\{1, 2, \ldots\}$, that is, without 0, but it is not an important distinction.)

Putting the initial conditions aside for a moment, for any recurrence, we can consider the subset $S$ of $V$ of solutions. For example, without initial conditions, in addition to the function $f$ given above, the Fibonacci relation is also solved by the function $g$ whose first few values are $g(0) = 1$, $g(1) = 1$, $g(2) = 3$, $g(3) = 4$, and $g(4) = 7$.

The subset $S$ is a subspace of $V$. It is nonempty because the zero function is a solution. It is closed under addition since if $f_1$ and $f_2$ are solutions, then

$$\begin{aligned}a_{n+1}(f_1 + f_2)(n+1) + \cdots + a_{n-k}(f_1 + f_2)(n-k) \\ &= (a_{n+1} f_1(n+1) + \cdots + a_{n-k} f_1(n-k)) \\ &\quad + (a_{n+1} f_2(n+1) + \cdots + a_{n-k} f_2(n-k)) \\ &= 0.\end{aligned}$$

And, it is closed under scalar multiplication since

$$\begin{aligned}
&a_{n+1}(rf_1)(n+1)+\cdots+a_{n-k}(rf_1)(n-k)\\
&\qquad\qquad = r(a_{n+1}f_1(n+1)+\cdots+a_{n-k}f_1(n-k))\\
&\qquad\qquad = r\cdot 0\\
&\qquad\qquad = 0.
\end{aligned}$$

We can give the dimension of $S$. Consider this map from the set of functions $S$ to the set of vectors $\mathbb{R}^k$.

$$f \mapsto \begin{pmatrix} f(0)\\ f(1)\\ \vdots\\ f(k)\end{pmatrix}$$

Exercise 3 shows that this map is linear. Because, as noted above, any solution of the recurrence is uniquely determined by the $k$ initial conditions, this map is one-to-one and onto. Thus it is an isomorphism, and thus $S$ has dimension $k$, the order of the recurrence.

So (again, without any initial conditions), we can describe the set of solutions of any linear homogeneous recurrence relation of degree $k$ by taking linear combinations of only $k$ linearly independent functions. It remains to produce those functions.

For that, we express the recurrence $f(n+1)=a_nf(n)+\cdots+a_{n-k}f(n-k)$ with a matrix equation.

$$\begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \ldots & a_{n-k+1} & a_{n-k}\\
1 & 0 & 0 & \ldots & 0 & 0\\
0 & 1 & 0 & & & \\
0 & 0 & 1 & & & \\
\vdots & \vdots & & \ddots & & \vdots\\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\begin{pmatrix} f(n)\\ f(n-1)\\ \vdots\\ f(n-k)\end{pmatrix}
=
\begin{pmatrix} f(n+1)\\ f(n)\\ \vdots\\ f(n-k+1)\end{pmatrix}$$

In trying to find the characteristic function of the matrix, we can see the pattern in the $2\times 2$ case

$$\begin{pmatrix} a_n-\lambda & a_{n-1}\\ 1 & -\lambda\end{pmatrix} = \lambda^2 - a_n\lambda - a_{n-1}$$

and $3\times 3$ case.

$$\begin{pmatrix} a_n-\lambda & a_{n-1} & a_{n-2}\\ 1 & -\lambda & 0\\ 0 & 1 & -\lambda\end{pmatrix} = -\lambda^3 + a_n\lambda^2 + a_{n-1}\lambda + a_{n-2}$$

Exercise 4 shows that the characteristic equation is this.

$$\begin{vmatrix} a_n-\lambda & a_{n-1} & a_{n-2} & \ldots & a_{n-k+1} & a_{n-k} \\ 1 & -\lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\lambda & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -\lambda \end{vmatrix}$$
$$= \pm(-\lambda^k + a_n\lambda^{k-1} + a_{n-1}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k})$$

We call that the polynomial 'associated' with the recurrence relation. (We will be finding the roots of this polynomial and so we can drop the $\pm$ as irrelevant.)

If $-\lambda^k + a_n\lambda^{k-1} + a_{n-1}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k}$ has no repeated roots then the matrix is diagonalizable and we can, in theory, get a formula for $f(n)$ as in the Fibonacci case. But, because we know that the subspace of solutions has dimension $k$, we do not need to do the diagonalization calculation, provided that we can exhibit $k$ linearly independent functions satisfying the relation.

Where $r_1, r_2, \ldots, r_k$ are the distinct roots, consider the functions $f_{r_1}(n) = r_1^n$ through $f_{r_k}(n) = r_k^n$ of powers of those roots. Exercise 5 shows that each is a solution of the recurrence and that the $k$ of them form a linearly independent set. So, given the homogeneous linear recurrence $f(n+1) = a_n f(n) + \cdots + a_{n-k}f(n-k)$ (that is, $0 = -f(n+1) + a_n f(n) + \cdots + a_{n-k}f(n-k)$) we consider the associated equation $0 = -\lambda^k + a_n\lambda^{k-1} + \cdots + a_{n-k+1}\lambda + a_{n-k}$. We find its roots $r_1, \ldots, r_k$, and if those roots are distinct then any solution of the relation has the form $f(n) = c_1r_1^n + c_2r_2^n + \cdots + c_kr_k^n$ for $c_1, \ldots, c_n \in \mathbb{R}$. (The case of repeated roots is also easily done, but we won't cover it here — see any text on Discrete Mathematics.)

Now, given some initial conditions, so that we are interested in a particular solution, we can solve for $c_1, \ldots, c_n$. For instance, the polynomial associated with the Fibonacci relation is $-\lambda^2 + \lambda + 1$, whose roots are $(1 \pm \sqrt{5})/2$ and so any solution of the Fibonacci equation has the form $f(n) = c_1((1+\sqrt{5})/2)^n + c_2((1-\sqrt{5})/2)^n$. Including the initial conditions for the cases $n = 0$ and $n = 1$ gives

$$\begin{aligned} c_1 + \phantom{(1-\sqrt{5}/2)}c_2 &= 1 \\ (1+\sqrt{5}/2)c_1 + (1-\sqrt{5}/2)c_2 &= 1 \end{aligned}$$

which yields $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, as was calculated above.

We close by considering the nonhomogeneous case, where the relation has the form $f(n+1) = a_n f(n) + a_{n-1}f(n-1) + \cdots + a_{n-k}f(n-k) + b$ for some nonzero $b$. As in the first chapter of this book, only a small adjustment is needed to make the transition from the homogeneous case. This classic example illustrates.

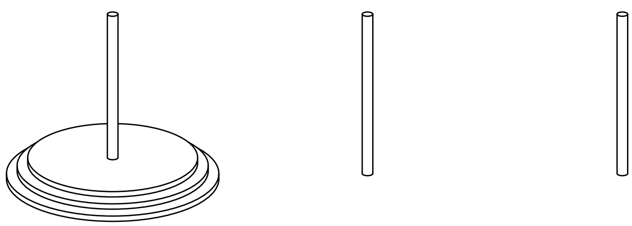
In 1883, Edouard Lucas posed the following problem.

> In the great temple at Benares, beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a

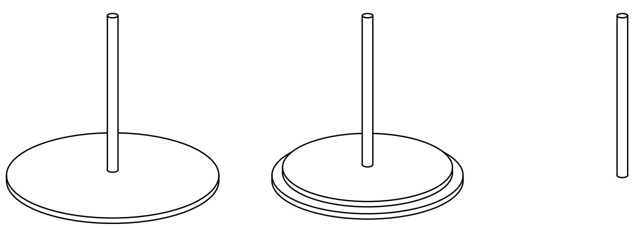
> bee. On one of these needles, at the creation, God placed sixty four disks of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Bramah. Day and night unceasingly the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Bramah, which require that the priest on duty must not move more than one disk at a time and that he must place this disk on a needle so that there is no smaller disk below it. When the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dusk, and with a thunderclap the world will vanish. (Translation of [De Parville] from [Ball & Coxeter].)

How many disk moves will it take? Instead of tackling the sixty four disk problem right away, we will consider the problem for smaller numbers of disks, starting with three.

To begin, all three disks are on the same needle.



After moving the small disk to the far needle, the mid-sized disk to the middle needle, and then moving the small disk to the middle needle we have this.



Now we can move the big disk over. Then, to finish, we repeat the process of moving the smaller disks, this time so that they end up on the third needle, on top of the big disk.

So the thing to see is that to move the very largest disk, the bottom disk, at a minimum we must: first move the smaller disks to the middle needle, then move the big one, and then move all the smaller ones from the middle needle to the ending needle. Those three steps give us this recurence.

$$T(n+1) = T(n) + 1 + T(n) = 2T(n) + 1 \quad \text{where } T(1) = 1$$

We can easily get the first few values of $T$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $T(n)$ | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | 1023 |

We recognize those as being simply one less than a power of two.

To derive this equation instead of just guessing at it, we write the original relation as $-1 = -T(n+1) + 2T(n)$, consider the homogeneous relation $0 = -T(n) + 2T(n-1)$, get its associated polynomial $-\lambda + 2$, which obviously has the single, unique, root of $r_1 = 2$, and conclude that functions satisfying the homogeneous relation take the form $T(n) = c_1 2^n$.

That's the homogeneous solution. Now we need a particular solution.

Because the nonhomogeneous relation $-1 = -T(n+1) + 2T(n)$ is so simple, in a few minutes (or by remembering the table) we can spot the particular solution $T(n) = -1$ (there are other particular solutions, but this one is easily spotted). So we have that — without yet considering the initial condition — any solution of $T(n+1) = 2T(n) + 1$ is the sum of the homogeneous solution and this particular solution: $T(n) = c_1 2^n - 1$.

The initial condition $T(1) = 1$ now gives that $c_1 = 1$, and we've gotten the formula that generates the table: the $n$-disk Tower of Hanoi problem requires a minimum of $2^n - 1$ moves.

Finding a particular solution in more complicated cases is, naturally, more complicated. A delightful and rewarding, but challenging, source on recurrence relations is [Graham, Knuth, Patashnik]., For more on the Tower of Hanoi, [Ball & Coxeter] or [Gardner 1957] are good starting points. So is [Hofstadter]. Some computer code for trying some recurrence relations follows the exercises.

## Exercises

**1** Solve each homogeneous linear recurrence relations.
  **(a)** $f(n+1) = 5f(n) - 6f(n-1)$
  **(b)** $f(n+1) = 4f(n-1)$
  **(c)** $f(n+1) = 6f(n) + 7f(n-1) + 6f(n-2)$

**2** Give a formula for the relations of the prior exercise, with these initial conditions.
  **(a)** $f(0) = 1$, $f(1) = 1$
  **(b)** $f(0) = 0$, $f(1) = 1$
  **(c)** $f(0) = 1$, $f(1) = 1$, $f(2) = 3$.

**3** Check that the isomorphism given betwween $S$ and $\mathbb{R}^k$ is a linear map. It is argued above that this map is one-to-one. What is its inverse?

**4** Show that the characteristic equation of the matrix is as stated, that is, is the polynomial associated with the relation. (Hint: expanding down the final column, and using induction will work.)

**5** Given a homogeneous linear recurrence relation $f(n+1) = a_n f(n) + \cdots + a_{n-k} f(n-k)$, let $r_1, \ldots, r_k$ be the roots of the associated polynomial.
  **(a)** Prove that each function $f_{r_i}(n) = r_k^n$ satisfies the recurrence (without initial conditions).
  **(b)** Prove that no $r_i$ is 0.
  **(c)** Prove that the set $\{f_{r_1}, \ldots, f_{r_k}\}$ is linearly independent.

**6** (This refers to the value $T(64) = 18,446,744,073,709,551,615$ given in the computer code below.) Transferring one disk per second, how many years would it take the priests at the Tower of Hanoi to finish the job?

### Computer Code

This code allows the generation of the first few values of a function defined by a recurrence and initial conditions. It is in the Scheme dialect of LISP (specifically, it was written for A. Jaffer's free scheme interpreter SCM, although it should run in any Scheme implementation).

First, the Tower of Hanoi code is a straightforward implementation of the recurrence.

```
(define (tower-of-hanoi-moves n)
    (if (= n 1)
       1
       (+ (* (tower-of-hanoi-moves (- n 1))
             2)
          1) )  )
```

(Note for readers unused to recursive code: to compute $T(64)$, the computer is told to compute $2 * T(63) - 1$, which requires, of course, computing $T(63)$. The computer puts the 'times 2' and the 'plus 1' aside for a moment to do that. It computes $T(63)$ by using this same piece of code (that's what 'recursive' means), and to do that is told to compute $2 * T(62) - 1$. This keeps up (the next step is to try to do $T(62)$ while the other arithmetic is held in waiting), until, after 63 steps, the computer tries to compute $T(1)$. It then returns $T(1) = 1$, which now means that the computation of $T(2)$ can proceed, etc., up until the original computation of $T(64)$ finishes.)

The next routine calculates a table of the first few values. (Some language notes: `'()` is the empty list, that is, the empty sequence, and `cons` pushes something onto the start of a list. Note that, in the last line, the procedure `proc` is called on argument `n`.)

```
(define (first-few-outputs proc n)
    (first-few-outputs-helper proc n '()) )
;
(define (first-few-outputs-aux proc n lst)
    (if (< n 1)
    lst
    (first-few-outputs-aux proc (- n 1) (cons (proc n) lst)) ) )
```

The session at the SCM prompt went like this.

```
>(first-few-outputs tower-of-hanoi-moves 64)
Evaluation took 120 mSec
(1 3 7 15 31 63 127 255 511 1023 2047 4095 8191 16383 32767
65535 131071 262143 524287 1048575 2097151 4194303 8388607
16777215 33554431 67108863 134217727 268435455 536870911
1073741823 2147483647 4294967295 8589934591 17179869183
34359738367 68719476735 137438953471 274877906943 549755813887
```

```
1099511627775 2199023255551 4398046511103 8796093022207
17592186044415 35184372088831 70368744177663 140737488355327
281474976710655 562949953421311 1125899906842623
2251799813685247 4503599627370495 9007199254740991
18014398509481983 36028797018963967 72057594037927935
144115188075855871 288230376151711743 576460752303423487
1152921504606846975 2305843009213693951 4611686018427387903
9223372036854775807 18446744073709551615)
```

This is a list of $T(1)$ through $T(64)$. (The 120 mSec came on a 50 mHz '486 running in an XTerm of XWindow under Linux. The session was edited to put line breaks between numbers.)

# Appendix

Mathematics is made of arguments (reasoned discourse that is, not crockery-throwing). This section is a reference to the most used techniques. A reader having trouble with, say, proof by contradiction, can turn here for an outline of that method.

But this section gives only a sketch. For more, these are classics: *Methods of Logic* by Quine, *Induction and Analogy in Mathematics* by Pólya, and *Naive Set Theory* by Halmos.

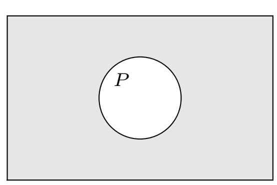
## IV.3 Propositions

The point at issue in an argument is the *proposition.* Mathematicians usually write the point in full before the proof and label it either *Theorem* for major points, *Corollary* for points that follow immediately from a prior one, or *Lemma* for results chiefly used to prove other results.

The statements expressing propositions can be complex, with many subparts. The truth or falsity of the entire proposition depends both on the truth value of the parts, and on the words used to assemble the statement from its parts.

**Not.** For example, where $P$ is a proposition, 'it is not the case that $P$' is true provided that $P$ is false. Thus, '$n$ is not prime' is true only when $n$ is the product of smaller integers.

We can picture the 'not' operation with a *Venn diagram.*

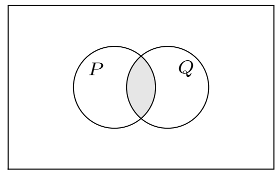



Where the box encloses all natural numbers, and inside the circle are the primes, the shaded area holds numbers satisfying 'not $P$'.

To prove that a 'not $P$' statement holds, show that $P$ is false.

**And.** Consider the statement form '$P$ and $Q$'. For the statement to be true both halves must hold: '7 is prime and so is 3' is true, while '7 is prime and 3 is not' is false.

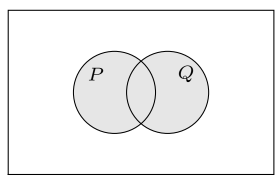
Here is the Venn diagram for '$P$ and $Q$'.




To prove '$P$ and $Q$', prove that each half holds.

**Or.** A '$P$ or $Q$' is true when either half holds: '7 is prime or 4 is prime' is true, while '7 is not prime or 4 is prime' is false. We take 'or' inclusively so that if both halves are true '7 is prime or 4 is not' then the statement as a whole is true. (In everyday speech, sometimes 'or' is meant in an exclusive way — "Eat your vegetables or no dessert" does not intend both halves to hold — but we will not use 'or' in that way.)

The Venn diagram for 'or' includes all of both circles.

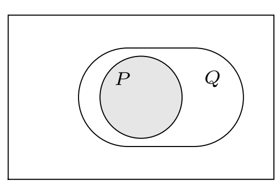



To prove '$P$ or $Q$', show that in all cases at least one half holds (perhaps sometimes one half and sometimes the other, but always at least one).

**If-then.** An 'if $P$ then $Q$' statement (sometimes written '$P$ materially implies $Q$' or just '$P$ implies $Q$' or '$P \implies Q$') is true unless $P$ is true while $Q$ is false. Thus 'if 7 is prime then 4 is not' is true while 'if 7 is prime then 4 is also prime' is false. (Contrary to its use in casual speech, in mathematics 'if $P$ then $Q$' does not connote that $P$ precedes $Q$ or causes $Q$.)

More subtly, in mathematics 'if $P$ then $Q$' is true when $P$ is false: 'if 4 is prime then 7 is prime' and 'if 4 is prime then 7 is not' are both true statements, sometimes said to be *vacuously true*. We adopt this convention because we want statements like 'if a number is a perfect square then it is not prime' to be true, for instance when the number is 5 or when the number is 6.

The diagram



shows that $Q$ holds whenever $P$ does (another phrasing is '$P$ is sufficient to give $Q$'). Notice again that if $P$ does not hold, $Q$ may or may not be in force.

There are two main ways to establish an implication. The first way is direct: assume that $P$ is true and, using that assumption, prove $Q$. For instance, to show 'if a number is divisible by 5 then twice that number is divisible by 10', assume that the number is $5n$ and deduce that $2(5n) = 10n$. The second way is indirect: prove the *contrapositive* statement: 'if $Q$ is false then $P$ is false' (rephrased, '$Q$ can only be false when $P$ is also false'). As an example, to show 'if a number is prime then it is not a perfect square', argue that if it were a square $p = n^2$ then it could be factored $p = n \cdot n$ where $n < p$ and so wouldn't be prime (of course $p = 0$ or $p = 1$ don't give $n < p$ but they are nonprime by definition).

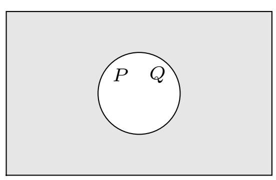
Note two things about this statement form.

First, an 'if $P$ then $Q$' result can sometimes be improved by weakening $P$ or strengthening $Q$. Thus, 'if a number is divisible by $p^2$ then its square is also divisible by $p^2$' could be upgraded either by relaxing its hypothesis: 'if a number is divisible by $p$ then its square is divisible by $p^2$', or by tightening its conclusion: 'if a number is divisible by $p^2$ then its square is divisible by $p^4$'.

Second, after showing 'if $P$ then $Q$', a good next step is to look into whether there are cases where $Q$ holds but $P$ does not. The idea is to better understand the relationship between $P$ and $Q$, with an eye toward strengthening the proposition.

**Equivalence.** An if-then statement cannot be improved when not only does $P$ imply $Q$, but also $Q$ implies $P$. Some ways to say this are: '$P$ if and only if $Q$', '$P$ iff $Q$', '$P$ and $Q$ are logically equivalent', '$P$ is necessary and sufficient to give $Q$', '$P \iff Q$'. For example, 'a number is divisible by a prime if and only if that number squared is divisible by the prime squared'.

The picture here shows that $P$ and $Q$ hold in exactly the same cases.




Although in simple arguments a chain like "$P$ if and only if $R$, which holds if and only if $S$ ..." may be practical, typically we show equivalence by showing the 'if $P$ then $Q$' and 'if $Q$ then $P$' halves separately.

## IV.4 Quantifiers

Compare these two statements about natural numbers: 'there is an $x$ such that $x$ is divisible by $x^2$' is true, while 'for all numbers $x$, that $x$ is divisible by $x^2$' is false. We call the 'there is' and 'for all' prefixes *quantifiers*.

**For all.** The 'for all' prefix is the *universal quantifier*, symbolized $\forall$.

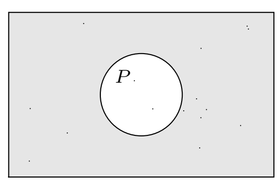
Venn diagrams aren't very helpful with quantifiers, but in a sense the box we draw to border the diagram shows the universal quantifier since it dilineates the universe of possible members.

To prove that a statement holds in all cases, we must show that it holds in each case. Thus, to prove 'every number divisible by $p$ has its square divisible by $p^2$', take a single number of the form $pn$ and square it $(pn)^2 = p^2n^2$. This is a "typical element" or "generic element" proof.

This kind of argument requires that we are careful to not assume properties for that element other than those in the hypothesis — for instance, this type of wrong argument is a common mistake: "if $n$ is divisible by a prime, say 2, so that $n = 2k$ then $n^2 = (2k)^2 = 4k^2$ and the square of the number is divisible by the square of the prime". That is an argument about the case $p = 2$, but it isn't a proof for general $p$.

**There exists.** We will also use the *existential quantifier*, symbolized $\exists$ and read 'there exists'.

As noted above, Venn diagrams are not much help with quantifiers, but a picture of 'there is a number such that $P$' would show both that there can be more than one and that not all numbers need satisfy $P$.




An existence proposition can be proved by producing something satisfying the property: once, to settle the question of primality of $2^{2^5}+1$, Euler produced its divisor 641. But there are proofs showing that something exists without saying how to find it; Euclid's argument given in the next subsection shows there are infinitely many primes without naming them. In general, while demonstrating existence is better than nothing, giving an example is better, and an exhaustive list of all instances is great. Still, mathematicians take what they can get.

Finally, along with "Are there any?" we often ask "How many?" That is why the issue of uniqueness often arises in conjunction with questions of existence. Many times the two arguments are simpler if separated, so note that just as proving something exists does not show it is unique, neither does proving something is unique show that it exists. (Obviously 'the natural number with more factors than any other' would be unique, but in fact no such number exists.)

## IV.5 Techniques of Proof

**Induction.** Many proofs are iterative, "Here's why the statement is true for for the case of the number 1, it then follows for 2, and from there to 3, and so on . . . ". These are called proofs by *induction.* Such a proof has two steps. In the *base step* the proposition is established for some first number, often 0 or 1. Then in the *inductive step* we assume that the proposition holds for numbers up to some $k$ and deduce that it then holds for the next number $k+1$.

Here is an example.

> We will prove that $1+2+3+\cdots+n = n(n+1)/2$.
>
> For the base step we must show that the formula holds when $n = 1$. That's easy, the sum of the first 1 number does indeed equal $1(1+1)/2$.
>
> For the inductive step, assume that the formula holds for the numbers $1, 2, \ldots, k$. That is, assume all of these instances of the formula.
>
> $$\begin{aligned} 1 &= 1(1+1)/2 \\ \text{and}\quad 1+2 &= 2(2+1)/2 \\ \text{and}\quad 1+2+3 &= 3(3+1)/2 \\ &\vdots \\ \text{and}\quad 1+\cdots+k &= k(k+1)/2 \end{aligned}$$
>
> From this assumption we will deduce that the formula therefore also holds in the $k+1$ next case. The deduction is straightforward algebra.
>
> $$1+2+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

We've shown in the base case that the above proposition holds for 1. We've shown in the inductive step that if it holds for the case of 1 then it also holds for 2; therefore it does hold for 2. We've also shown in the inductive step that if the statement holds for the cases of 1 and 2 then it also holds for the next case 3, etc. Thus it holds for any natural number greater than or equal to 1.

Here is another example.

> We will prove that every integer greater than 1 is a product of primes.
>
> The base step is easy: 2 is the product of a single prime.
>
> For the inductive step assume that each of $2, 3, \ldots, k$ is a product of primes, aiming to show $k+1$ is also a product of primes. There are two

> possibilities: (i) if $k+1$ is not divisible by a number smaller than itself then it is a prime and so is the product of primes, and (ii) if $k+1$ is divisible then its factors can be written as a product of primes (by the inductive hypothesis) and so $k+1$ can be rewritten as a product of primes. That ends the proof.
>
> (*Remark.* The Prime Factorization Theorem of Number Theory says that not only does a factorization exist, but that it is unique. We've shown the easy half.)

There are two things to note about the 'next number' in an induction argument.

For one thing, while induction works on the integers, it's no good on the reals. There is no 'next' real.

The other thing is that we sometimes use induction to go down, say, from 10 to 9 to 8, etc., down to 0. So 'next number' could mean 'next lowest number'. Of course, at the end we have not shown the fact for all natural numbers, only for those less than or equal to 10.

**Contradiction.** Another technique of proof is to show something is true by showing it can't be false.

> The classic example is Euclid's, that there are infinitely many primes.
>
> Suppose there are only finitely many primes $p_1, \ldots, p_k$. Consider $p_1 \cdot p_2 \ldots p_k + 1$. None of the primes on this supposedly exhaustive list divides that number evenly, each leaves a remainder of 1. But every number is a product of primes so this can't be. Thus there cannot be only finitely many primes.

Every proof by contradiction has the same form: assume that the proposition is false and derive some contradiction to known facts.

> Another example is this proof that $\sqrt{2}$ is not a rational number.
>
> Suppose that $\sqrt{2} = m/n$.
>
> $$2n^2 = m^2$$
>
> Factor out the 2's: $n = 2^{k_n} \cdot \hat{n}$ and $m = 2^{k_m} \cdot \hat{m}$ and rewrite.
>
> $$2 \cdot (2^{k_n} \cdot \hat{n})^2 = (2^{k_m} \cdot \hat{m})^2$$
>
> The Prime Factorization Theorem says that there must be the same number of factors of 2 on both sides, but there are an odd number $1 + 2k_n$ on the left and an even number $2k_m$ on the right. That's a contradiction, so a rational with a square of 2 cannot be.

Both of these examples aimed to prove something doesn't exist. A negative proposition often suggests a proof by contradiction.

## IV.6 Sets, Functions, and Relations

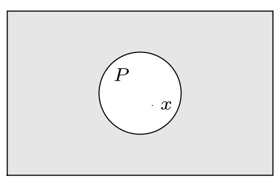
**Sets.** Mathematicians work with collections called *sets*. A set can be given as a listing between curly braces as in $\{1, 4, 9, 16\}$, or, if that's unwieldy, by using set-builder notation as in $\{x \mid x^5 - 3x^3 + 2 = 0\}$ (read "the set of all $x$ such that $\ldots$"). We name sets with capital roman letters as with the primes $P = \{2, 3, 5, 7, 11, \ldots\}$, except for a few special sets such as the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. To denote that something is an *element* (or *member*) of a set we use '$\in$', so that $7 \in \{3, 5, 7\}$ while $8 \notin \{3, 5, 7\}$.

What distinguishes a set from any other type of collection is the Principle of Extensionality, that two sets with the same elements are equal. Because of this principle, in a set repeats collapse $\{7, 7\} = \{7\}$ and order doesn't matter $\{2, \pi\} = \{\pi, 2\}$.

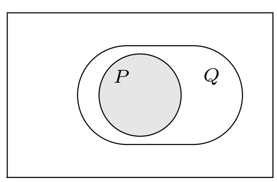
We use '$\subset$' for the subset relationship: $\{2, \pi\} \subset \{2, \pi, 7\}$ and '$\subseteq$' for subset or equality (if $A$ is a subset of $B$ but $A \neq B$ then $A$ is a *proper subset* of $B$). These symbols may be flipped, for instance $\{2, \pi, 5\} \supset \{2, 5\}$.

Because of Extensionality, to prove that two sets are equal $A = B$, just show that they have the same members. Usually we show mutual inclusion, that both $A \subseteq B$ and $A \supseteq B$.

**Set operations.** Venn diagrams are handy here. For instance, $x \in P$ can be pictured

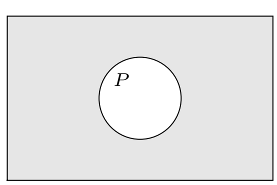



and '$P \subseteq Q$' looks like this.

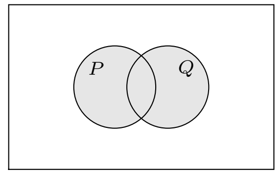



Note that this is a repeat of the diagram for 'if … then … ' propositions. That's because '$P \subseteq Q$' means 'if $x \in P$ then $x \in Q$'.

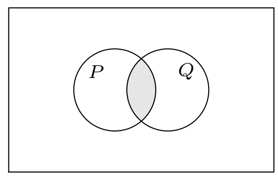
In general, for every propositional logic operator there is an associated set operator. For instance, the *complement* of $P$ is $P^{\text{comp}} = \{x \mid \text{not}(x \in P)\}$



the *union* is $P \cup Q = \{x \bigm| (x \in P) \text{ or } (x \in Q)\}$




and the *intersection* is $P \cap Q = \{x \bigm| (x \in P) \text{ and } (x \in Q)\}$.




When two sets share no members their intersection is the *empty set* $\{\}$, symbolized $\varnothing$. Any set has the empty set for a subset, by the 'vacuously true' property of the definition of implication.

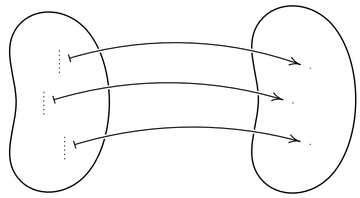
**Sequences.** We shall also use collections where order does matter and where repeats do not collapse. These are *sequences*, denoted with angle brackets: $\langle 2,3,7\rangle \neq \langle 2,7,3\rangle$. A sequence of length 2 is sometimes called an *ordered pair* and written with parentheses: $(\pi, 3)$. We also sometimes say 'ordered triple', 'ordered 4-tuple', etc. The set of ordered $n$-tuples of elements of a set $A$ is denoted $A^n$. Thus the set of pairs of reals is $\mathbb{R}^2$.

**Functions.** We first see functions in elementary Algebra, where they are presented as formulas (e.g., $f(x) = 16x^2 - 100$), but progressing to more advanced Mathematics reveals more general functions — trigonometric ones, exponential and logarithmic ones, and even constructs like absolute value that involve piecing together parts — and we see that functions aren't formulas, instead the key idea is that a function associates with its input $x$ a single output $f(x)$.

Consequently, a *function* or *map* is defined to be a set of ordered pairs $(x, f(x)\,)$ such that $x$ suffices to determine $f(x)$, that is: if $x_1 = x_2$ then $f(x_1) = f(x_2)$ (this requirement is referred to by saying a function is *well-defined*).*

Each input $x$ is one of the function's *arguments* and each output $f(x)$ is a *value*. The set of all arguments is $f$'s *domain* and the set of output values is its *range*. Usually we don't need know what is and is not in the range and we instead work with a superset of the range, the *codomain*. The notation for a function $f$ with domain $X$ and codomain $Y$ is $f\colon X \to Y$.

*More on this is in the section on isomorphisms



We sometimes instead use the notation $x \overset{f}{\longmapsto} 16x^2 - 100$, read '$x$ maps under $f$ to $16x^2 - 100$', or '$16x^2 - 100$ is the *image* of $x$'.

Some maps, like $x \mapsto \sin(1/x)$, can be thought of as combinations of simple maps, here, $g(y) = \sin(y)$ applied to the image of $f(x) = 1/x$. The *composition* of $g\colon Y \to Z$ with $f\colon X \to Y$, is the map sending $x \in X$ to $g(\,f(x)\,) \in Z$. It is denoted $g \circ f\colon X \to Z$. This definition only makes sense if the range of $f$ is a subset of the domain of $g$.

Observe that the *identity map* $\mathrm{id}\colon Y \to Y$ defined by $\mathrm{id}(y) = y$ has the property that for any $f\colon X \to Y$, the composition $\mathrm{id} \circ f$ is equal to $f$. So an identity map plays the same role with respect to function composition that the number 0 plays in real number addition, or that the number 1 plays in multiplication.

In line with that analogy, define a *left inverse* of a map $f\colon X \to Y$ to be a function $g\colon \mathrm{range}(f) \to X$ such that $g \circ f$ is the identity map on $X$. Of course, a *right inverse* of $f$ is a $h\colon Y \to X$ such that $f \circ h$ is the identity.

A map that is both a left and right inverse of $f$ is called simply an *inverse*. An inverse, if one exists, is unique because if both $g_1$ and $g_2$ are inverses of $f$ then $g_1(x) = g_1 \circ (f \circ g_2)(x) = (g_1 \circ f) \circ g_2(x) = g_2(x)$ (the middle equality comes from the associativity of function composition), so we often call it "the" inverse, written $f^{-1}$. For instance, the inverse of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 3$ is the function $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = (x + 3)/2$.

The superscript '$f^{-1}$' notation for function inverse can be confusing — it doesn't mean $1/f(x)$. It is used because it fits into a larger scheme. Functions that have the same codomain as domain can be iterated, so that where $f\colon X \to X$, we can consider the composition of $f$ with itself: $f \circ f$, and $f \circ f \circ f$, etc. Naturally enough, we write $f \circ f$ as $f^2$ and $f \circ f \circ f$ as $f^3$, etc. Note that the familiar exponent rules for real numbers obviously hold: $f^i \circ f^j = f^{i+j}$ and $(f^i)^j = f^{i \cdot j}$. The relationship with the prior paragraph is that, where $f$ is invertible, writing $f^{-1}$ for the inverse and $f^{-2}$ for the inverse of $f^2$, etc., gives that these familiar exponent rules continue to hold, once $f^0$ is defined to be the identity map.

If the codomain $Y$ equals the range of $f$ then we say that the function is *onto*. A function has a right inverse if and only if it is onto (this is not hard to check). If no two arguments share an image, if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$, then the function is *one-to-one*. A function has a left inverse if and only if it is one-to-one (this is also not hard to check).

By the prior paragraph, a map has an inverse if and only if it is both onto and one-to-one; such a function is a *correspondence*. It associates one and only one element of the domain with each element of the range (for example, finite sets must have the same number of elements to be matched up in this way). Because a composition of one-to-one maps is one-to-one, and a composition of onto maps is onto, a composition of correspondences is a correspondence.

We sometimes want to shrink the domain of a function. For instance, we may take the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and, in order to have an inverse, limit input arguments to nonnegative reals $\hat{f}\colon \mathbb{R}^+ \to \mathbb{R}$. Technically, $\hat{f}$ is a different function than $f$; we call it the *restriction* of $f$ to the smaller domain.

A final point on functions: neither $x$ nor $f(x)$ need be a number. As an example, we can think of $f(x, y) = x + y$ as a function that takes the ordered pair $(x, y)$ as its argument.

**Relations.** Some familiar operations are obviously functions: addition maps $(5, 3)$ to 8. But what of '$<$' or '$=$'? We here take the approach of rephrasing '$3 < 5$' to '$(3, 5)$ is in the relation $<$'. That is, define a *binary relation* on a set $A$ to be a set of ordered pairs of elements of $A$. For example, the $<$ relation is the set $\{(a, b) \bigm| a < b\}$; some elements of that set are $(3, 5)$, $(3, 7)$, and $(1, 100)$.

Another binary relation on the natural numbers is equality; this relation is formally written as the set $\{\ldots, (-1, -1), (0, 0), (1, 1), \ldots\}$.

Still another example is 'closer than 10', the set $\{(x, y) \bigm| |x - y| < 10\}$. Some members of that relation are $(1, 10)$, $(10, 1)$, and $(42, 44)$. Neither $(11, 1)$ nor $(1, 11)$ is a member.

Those examples illustrate the generality of the definition. All kinds of relationships (e.g., 'both numbers even' or 'first number is the second with the digits reversed') are covered under the definition.

**Equivalence Relations.** We shall need to say, formally, that two objects are alike in some way. While these alike things aren't identical, they are related (e.g., two integers that 'give the same remainder when divided by 2').

A binary relation $\{(a, b), \ldots\}$ is an *equivalence relation* when it satisfies

(1) *reflexivity*: any object is related to itself;

(2) *symmetry*: if $a$ is related to $b$ then $b$ is related to $a$;

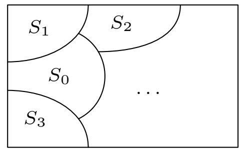
(3) *transitivity*: if $a$ is related to $b$ and $b$ is related to $c$ then $a$ is related to $c$.

(To see that these conditions formalize being the same, read them again, replacing 'is related to' with 'is like'.)

Some examples (on the integers): '$=$' is an equivalence relation, '$<$' does not satisfy symmetry, 'same sign' is a equivalence, while 'nearer than 10' fails transitivity.

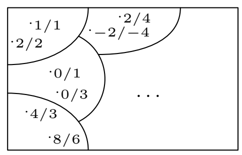
**Partitions.** In 'same sign' $\{(1, 3), (-5, -7), (-1, -1), \ldots\}$ there are two kinds of pairs, the first with both numbers positive and the second with both negative. So integers fall into exactly one of two classes, positive or negative.

A *partition* of a set $S$ is a collection of subsets $\{S_1, S_2, \ldots\}$ such that every element of $S$ is in one and only one $S_i$: $S_1 \cup S_2 \cup \ldots = S$, and if $i$ is not equal to $j$ then $S_i \cap S_j = \varnothing$. Picture $S$ being decomposed into distinct parts.




Thus, the first paragraph says 'same sign' partitions the integers into the positives and the negatives. Similarly, the equivalence relation '=' partitions the integers into one-element sets.

Another example is the fractions. Of course, $2/3$ and $4/6$ are equivalent fractions. That is, for the set $S = \{n/d \bigm| n, d \in \mathbb{Z} \text{ and } d \neq 0\}$, we define two elements $n_1/d_1$ and $n_2/d_2$ to be equivalent if $n_1 d_2 = n_2 d_1$. We can check that this is an equivalence relation, that is, that it satisfies the above three conditions. With that, $S$ is divided up into parts.




Before we show that equivalence relations always give rise to partitions, we first illustrate the argument. Consider the relationship between two integers of 'same parity', the set $\{(-1, 3), (2, 4), (0, 0), \ldots\}$ (i.e., 'give the same remainder when divided by 2'). We want to say that the natural numbers split into two pieces, the evens and the odds, and inside a piece each member has the same parity as each other. So for each $x$ we define the set of numbers associated with it: $S_x = \{y \bigm| (x, y) \in \text{'same parity'}\}$. Some examples are $S_1 = \{\ldots, -3, -1, 1, 3, \ldots\}$, and $S_4 = \{\ldots, -2, 0, 2, 4, \ldots\}$, and $S_{-1} = \{\ldots, -3, -1, 1, 3, \ldots\}$. These are the parts, e.g., $S_1$ is the odds.

**Theorem.** An equivalence relation induces a partition on the underlying set.

Proof. Call the set $S$ and the relation $R$. In line with the illustration in the paragraph above, for each $x \in S$ define $S_x = \{y \bigm| (x, y) \in R\}$.

Observe that, as $x$ is a member if $S_x$, the union of all these sets is $S$. So we will be done if we show that distinct parts are disjoint: if $S_x \neq S_y$ then $S_x \cap S_y = \varnothing$. We will verify this through the contrapositive, that is, we wlll assume that $S_x \cap S_y \neq \varnothing$ in order to deduce that $S_x = S_y$.

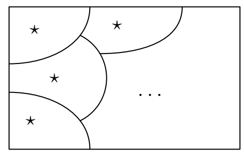
Let $p$ be an element of the intersection. Then by definition of $S_x$ and $S_y$, the two $(x, p)$ and $(y, p)$ are members of $R$, and by symmetry of this relation $(p, x)$ and $(p, y)$ are also members of $R$. To show that $S_x = S_y$ we will show each is a subset of the other.

Assume that $q \in S_x$ so that $(q, x) \in R$. Use transitivity along with $(x, p) \in R$ to conclude that $(q, p)$ is also an element of $R$. But $(p, y) \in R$ so another use of transitivity gives that $(q, y) \in R$. Thus $q \in S_y$. Therefore $q \in S_x$ implies $q \in S_y$, and so $S_x \subseteq S_y$.

The same argument in the other direction gives the other inclusion, and so the two sets are equal, completing the contrapositive argument. QED

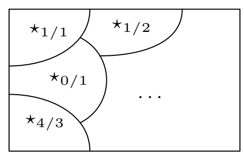
We call each part of a partition an *equivalence class* (or informally, 'part').

We somtimes pick a single element of each equivalence class to be the *class representative*.



Usually when we pick representatives we have some natural scheme in mind. In that case we call them the *canonical* representatives.

An example is the simplest form of a fraction. We've defined 3/5 and 9/15 to be equivalent fractions. In everyday work we often use the 'simplest form' or 'reduced form' fraction as the class representatives.



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