A Brief Guide to Mathematical Writing

In addition to having strong competency in mathematics, a mathematics teacher must be able to communicate mathematics both orally and in writing. This document gives some basic rules to follow when writing mathematics. These rules apply to both the writing of proofs as well as to general exposition in mathematics.

1. Know your audience

   Writing something for a fellow teacher or a journal will most likely require a different style than writing for your students. A basic rule of thumb here is to include enough explanation so that the intended reader can follow what you are saying. However, you don't need to include every detail all the time. There is nothing wrong with expecting the reader to be actively involved with the reading. Just don't skip a detail which requires the reader to spend hours filling in. This applies in particular to assignments for the course.

2. Give yourself time to do the writing.

   Whether it is an assignment for a class you are taking or a handout for your own students, don't try to put it together a few minutes before you need to have it done. This will give you time to read over what you have written to see if it makes sense. Something which sounds good in your head doesn't always sound good on paper. You will also have time to see that everything is spelled correctly, if you have used the right words, used good grammar, and so forth. It is hard enough to write well when you have time, even harder when you rush.

3. Keep the reader informed

   This is especially important when writing proofs.

   - Tell the reader what you plan to do, then do it. For instance, when proving a proposition state the assumptions and identify the proof technique you are going to use. For instance:

     Proposition: If x is an odd integer, then x^2 is an odd integer.
     Proof : Assume that x is an odd integer. We will give a direct proof that x^2 is an odd integer.

   - If you are using a previous proposition as a reason in a proof, point this out to the reader. If the proposition has a specific name, then use it. For example:

     By the Pythagorean Theorem, we have that ...

   - If you want to use a proposition which does not have a name, try to indicate the content of the proposition. For instance:

     By a previous result about even numbers we can say ...

   - If the reader doesn't recognize the result by the way you have applied it, at least they have been given an idea about where to look it up.
   - If you use a symbol or term that is new or not used much, make certain to define it for the reader or to recall what it means.
   - Try not to put up roadblocks which cause the reader to take unnecessary detours. For example, if you write:

But n is a prime number. To see this, consider ...

This might cause the reader to stop and look back to see why n is a prime number. They may not notice that you intend to prove it. One suggestion here would be to say:

Now (or Next) we show that n is a prime number.

- Tell the reader when the proof is finished. This can be done by using a symbol such as $\square$, using the letters QED which stand for "quod erat demonstrandum (That which was proved), or using words such as: *This completes the proof.*
- Be careful when saying "clearly" or that something is "obvious". If you say something is "obvious", that means that a person in the intended audience could figure it out in very little time and with very little effort.

4. Make certain that mathematical symbols stand out from the prose in your writing

- Use italics for variables, equations, names of sets, and so on.
- Mathematical expressions should be displayed on separate lines and centered on the page. The same holds if you are going to do a computation. For instance, you should not write:

$$xy = (2n+1)(2m+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Instead, you should write:

$$\begin{aligned} xy &= (2n+1)(2m+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2nm + m + n) + 1 \end{aligned}$$

5. Do not begin a sentence with a mathematical symbol

For instance do not write:

$n \in \mathbf{Z}$, so we have ...

You should write something like:

If $n \in \mathbf{Z}$, then we have ...

6. Separate symbols not in a list by words, if possible

- Separating symbols by words makes the statement easier to read and so easier to understand. For example, the sentence:

Except for a, b is the only root of the equation $(x-a)(x-b) = 0$,

would be easier to understand if written as:

Except for a, the number b is the only root of the equation $(x-a)(x-b) = 0$.

- It is alright to use commas to separate symbols in a list, such as:

Let $a, b, c \in \mathbf{Z}$ or Suppose A, B, C are sets.

7. Use the pronoun "we" instead of "I"

   This is especially true when writing a proof. The idea is that the reader and the writer are proving the proposition together. Also, using "I" a lot can begin to sound conceited. For example, you should say:

   We will now show that G is a group or Let's show that G is a group.

8. Avoid using informal and colloquial language as much as possible

   - This means that like you should not like, you know, write like you talk. Get my drift, man? (Dude?)
   - It is incorrect to say *Next, plug 7 in for x.* You should say *Next, substitute 7 for x.*
   - Avoid foul or suggestive language.

9. Be familiar with common words and phrases used in mathematics

   - The words "each" and "every" can be used interchangeably. For instance, the following statements say the same thing:

     For each $n \in \mathbf{Z}$, we know that $n^2 \geq 0$.

     For every $n \in \mathbf{Z}$, we know that $n^2 \geq 0$.

     They are saying no matter integer n we choose, n^2 is always greater than or equal to zero. If there were just one integer n for which n^2 is less than zero, then we cannot use "each" or "every".
   - The words and phrases "therefore", "thus", "hence", "consequently", "so", "it follows that","this implies that" all have the same meaning. These are indispensable words when writing mathematics. As such, you should not always use the same word over and over again. Introduce a little variety into your writing.
   - Be careful of the phrase "Since ..., then ...". Even though many people connect these two words, it just doesn't sound right. For instance, consider the statement:

     Since n^2 is even, then n is even.

     Somehow, it just sounds awkward. In fact, in situations like this, you can replace "since" with "because". Other better sounding versions of the statement are:

     If n^2 is even, then n is even.

     Since n^2 is even, it follows that n is even.

     Since n^2 is even, n is even.

10. When using a number as part of an English sentence, try to use the word for the number, not the numeral

    - For instance, don't write

      The difference between x and y is 0.

Instead write,

The difference between x and y is zero.

- More generally, don't mix words with symbols improperly. Instead of writing

  Every integer ≥ 2 is a prime or is composite,

  it is better to write

  Every integer greater than or equal to two is a prime or is composite

  or

  If $n \geq 2$ is an integer, then n is prime for composite.

- Avoid using a symbol in any general statement, like the statement of a proposition, when it is not necessary. For example, try not to write

  Every nonzero real number x has an inverse.

  Including the symbol x serves no useful purpose. You can delete it and the statement still makes sense.

- If the symbol x is being used as part of a proof, then we can write

  The nonzero real number x has an inverse.

11. More on using symbols

- Explain the meaning of every symbol you use. Symbols that have a "fixed" meaning, such as $\mathbf{Z}$ or $\cup$, only have to be defined the very first time they are used. Try not to use a fixed symbol in any way other than the way it is commonly used.
- Try to be consistent in our use of symbols. Don't accidently change notation for a quantity halfway through a proof. Also, if you are defining related quantities, use letters in alphabetical order or subscripts. For instance you can say
- Choose symbols that are consistent with mathematical convention. There are no hard and fast rules, but if you look for it, you will notice a remarkable consistency. For example, real number variables are usually denoted by u, v, x, and y; real number constants are often denoted by a, b, and c; integers and natural number are typically denoted by k, n, m, p, and q; and functions are often named f, g, or h, with p and q often reserved for polynomial functions. The best way to learn the conventions is to pay attention to how symbols are used in class and in your reading.

12. Try to avoid the use of "passive voice".

13. Grammar and spelling

Make every effort to use good grammar, good spelling, and good organization in your writing. Don't jump around in your proofs. Write them so that they flow in a logical manner. You shouldn't be using phrases like, "I forgot to mention that ...", or something similar. If you get to a stage in a proof and you realize that something should have been mentioned earlier, then rewrite the proof in the proper order.

14. A Simple Example To put this all together

Suppose we want to prove that when the product of two numbers is odd, then the numbers are odd.

This is easily done by contraposition. That is, it is easy to prove the logically equivalent statement that if at least one of two numbers is even, then their product is even.

Note that while most readers would understand from the context that the numbers in question have to be integers, we can improve the clarity by specifying that they are integers. Since it is easier to discuss two numbers if they have names, we introduce notation to assist us, calling the numbers a and b. We put this together in a proof as follows:

Theorem: If the Product of two numbers is odd, then the numbers are both odd.

Proof. We prove the contrapositive. Assume that a and b are integers and at least one of them is even. We will show that ab is even. Since the names are arbitrary, we assume without loss of generality that a is even. Therefore, there is an integer c satisfying $a = 2c$. Hence, $ab = 2cb = 2(cb)$. Since cb is an integer, the product ab is even, and we are done.

Note that we began the proof by identifying the proof technique we will use. We then established a notation, stated our assumptions, and clearly identified the statement we intend to prove. The proof itself is straightforward, relying only on the definition of even.

Note that we specified that c is an integer when it was introduced. We clearly indicated when the proof was completed, which is an invitation to our reader to go back and check that we actually proved what we set out to prove.

15. A Checklist for Evaluating Homework Write-ups

Before turning in your homework assignment, ask yourself the following questions about each problem:

- Have you clearly (re)stated the problem to be solved?
- Is your solution self-contained?
- Have you included an appropriate amount of detail for your intended reader?
- Is the solution written in complete sentences and with appropriate attention to grammar and spelling?
- Have you used paragraphs where necessary?
- Have you included remarks in proofs to help the reader understand your reasoning?
- Did you clearly label any diagrams, tables, or graphs used?
- Are all variables and symbols explicitly defined?
- Are all variables and symbols necessary or could some be eliminated?
- Have you explicitly referenced all definitions, theorems, and formulas used?
- Did you give acknowledgment where it is due?
- Did you answer the question that was asked?
- Have you checked that the mathematics is correct?